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Abstract. We introduce cooperative TU-games on concept lattices, where a concept is a pair (S, S') with S being a subset of players or objects, and S' a subset of attributes. Any such game induces a game on the set of players/objects, which appears to be a TU-game whose collection of feasible coalitions is a lattice closed under intersection, and a game on the set of attributes. We propose a Shapley value for each type of game, axiomatize it, and investigate the geometrical properties of the core (non-emptiness, boundedness, pointedness, extremal rays). In particular, we derive the equivalence of the intent and extent core for the class of distributive concepts.

Keywords: cooperative game, restricted cooperation, concept lattice, core, Shapley value

1 Introduction

Cooperative games with transferable utility (TU-games) have been widely studied and used in many domains of applications. N being a set of players, or more generally, a set of abstract objects, a TU-game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ assigns to every coalition or group $S \subseteq N$ a number representing its "worth" (monetary value: benefit created by the cooperation of the members of S, or cost saved by the common usage of a service by the members of S, power, importance, etc.).

Once the function v is determined, the main concern of cooperative game theory is to provide a rational scheme for distributing the total worth $v(N)$ of the cooperation among the members of N (or determining individual power/importance degrees, if $v(N)$ is not interpreted as a monetary value). The until now most popular methods to achieve this are the Shapley value [19] and the core [15]. The Shapley value yields a single distribution vector, satisfying a set of four natural axioms (Pareto optimality, symmetry, linearity, null player property), while the core is a set of distribution vectors that are Pareto optimal and satisfy coalitional rationality (i.e., a coalition receives at least its own worth). While the Shapley value always exists for any game, the core is a convex polyhedron, but may be empty.

In many situations, however, not all subsets of N can be realized as coalitions or are feasible, which means that the mapping v is defined on a subcollection $\mathcal{F}$ of 2^N

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only. Pioneering works considering this situation are due to Aumann and Drèze [3], who speak of *coalition structure*, and later to Faigle and Kern [10], who speak of *restricted cooperation*. $\mathcal{F}$ has been studied under many structural assumptions, such as distributive lattices (closed under union and intersection) [13], convex geometries [4, 5], antimatroids [2], union-stable systems [1] (a.k.a. weakly union-closed systems [12, 11]), etc. In this case, the study of the geometric properties of the core is challenging since the core may become unbounded or have no vertices (see a survey in [16]). Also the Shapley value has to be redefined, and its axiomatization may become difficult.

In many cases, the structural assumptions on $\mathcal{F}$ are not clearly motivated or are too restrictive. The aim of this paper is to study a structure for $\mathcal{F}$ which is both fairly general (a lattice of sets closed under intersection), and produced in a natural way, through a set of attributes possessed by the players or objects in N[1]. In short, our framework is based on *concept lattices* [6, 7, 17], a notion which has lead to the now quite active field of *formal concept analysis* [14]. M being a set of attributes, a concept is a pair (S, S') with $S \subseteq N$ and $S' \subseteq M$ such that S' is the set of those attributes that are satisfied by all members of S. A remarkable result is that any (finite) lattice is isomorphic to a concept lattice, and that the lattice of extents (i.e., the lattice of concepts (S, S') limited to the first arguments S) is a set lattice closed under intersection, and moreover any such lattice arises that way. We define a game v on the lattice of concepts, dividing it into a game v_N on the lattice of extents (which corresponds to a game with restricted cooperation $(\mathcal{F}, v)$ where $\mathcal{F}$ is a lattice closed under intersection), and a game v_M on the lattice of intents (which corresponds to a game on the set of attributes). For both types of games, we propose a Shapley value with its axiomatization. Moreover, we investigate in details the properties of the core. Our results can be seen to generalize many results of the literature on games with restricted cooperation.

The paper is organized as follows. Section 2 introduces the main notions needed in the paper: cooperative games, concept lattices and games on concept lattices. Section 3 proposes a definition for the Shapley value, which is a natural generalization of those values presented by Faigle and Kern [13], and Bilbao and Edelman [5], together with its axiomatization. Section 4 studies the properties of the core: nonemptiness, boundedness, pointedness, and extremal rays. Some interesting properties of balanced collections are also presented.

2 Framework

2.1 Cooperative games

Let $N = \{1, \dots, n\}$ be a finite set of players. A *cooperative (TU) game* (or *game* for short) on N is a mapping $v : 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. Any subset $S \subseteq N$ is called a *coalition*. The quantity $v(S)$ represents the "worth" of the coalition, that is, depending on the application context, the benefit realized (or cost saved, etc.) by cooperation of the members of S.

[1] We do not claim for full generality, since there remain important cases which are not covered by our model. For instance, games on communication networks introduced by Myerson [18] are defined on the set of connected subsets, which are not closed under intersection in general.

We consider the general case where the cooperation is restricted, i.e., where the set $\mathcal{F}$ of all feasible coalitions might be a proper subset of 2^N. We denote the corresponding *game with restricted cooperation* as a pair $(\mathcal{F}, v)$, or simply v if there is no ambiguity.

Let us consider a cooperative game $(\mathcal{F}, v)$ with $N \in \mathcal{F}$. A payoff vector is a vector $x \in \mathbb{R}^n$. For any $S \subseteq N$, we denote by $x(S) = \sum_{i \in S} x_i$ the total payoff given by x to the coalition S. The payoff vector x is *efficient* if $x(N) = v(N)$. The core of a cooperative game is the set of efficient payoff vectors such that no coalition can achieve a better payoff by itself:

$$\text{core}(\mathcal{F}, v) = \{x \in \mathbb{R}^n \mid x(S) \geq v(S)\ \forall S \in \mathcal{F},\ \text{and } x(N) = v(N)\}.$$

Note that $\text{core}(\mathcal{F}, v)$ is a convex closed bounded polyhedron when $\mathcal{F} = 2^N$. In other cases, the core may be unbounded or non pointed, and its study becomes difficult (see [10] and a survey in [16]). We recall from the theory of polyhedra that a polyhedron defined by a set of inequalities $\mathbf{Ax} \geq \mathbf{b}$ is the Minkowski sum of its convex part and its conic part (the so-called recession cone), the latter being determined by the inequalities $\mathbf{Ax} \geq \mathbf{0}$, and being therefore independent of the righthand side $\mathbf{b}$. So the recession cone of $\text{core}(\mathcal{F}, v)$ is the polyhedron $\text{core}(\mathcal{F}, 0)$, which does not depend on v.

A collection $\mathcal{B} \subseteq \mathcal{F}$ of nonempty sets is said to be *balanced* if there exist positive weights $\lambda_S, S \in \mathcal{B}$ such that

$$\sum_{S \in \mathcal{B}, S \ni i} \lambda_S = 1 \quad \forall i \in N.$$

A game $(\mathcal{F}, v)$ is said to be *balanced* if $v(N) \geq \sum_{S \in \mathcal{B}} \lambda_S v(S)$ holds for every balanced collection $\mathcal{B}$ with weight system $(\lambda_S)_{S \in \mathcal{B}}$. It is well-known that the core of v is nonempty if and only if v is balanced [10].

2.2 Concept lattices

We begin by recalling that a *lattice* is a partially ordered set (poset) $(L, \preceq)$, where $\preceq$ is reflexive, antisymmetric and transitive, such that for any two elements $x, y \in L$, a supremum $x \vee y$ and an infimum $x \wedge y$ exist. If no ambiguity occurs, the lattice is simply denoted by L. The *dual partial order* $\preceq^\partial$ is defined by $x \preceq^\partial y$ if and only if $y \preceq x$. The *dual* of the lattice $(L, \preceq)$ is the poset $(L, \preceq^\partial)$, denoted by L^∂ if no ambiguity occurs.

A *context* (see, e.g., [6, 7, 14, 17]) is a triple $\mathcal{C} = (N, M, I)$, where N is a finite nonempty set of objects, M is a finite set of attributes, and $I : N \times M \to \{0, 1\}$ is a binary relation defined by $I(i, a) = 1$ if object $i \in N$ satisfies attribute $a \in M$, and 0 otherwise. The binary relation can be represented as a matrix or table called the *incidence matrix (table).*

Let $\mathcal{C} = (N, M, I)$ be a context. The *intent* of a subset of objects $S \subseteq N$ is defined as the set of attributes satisfied by all objects in S:

$$S'_\mathcal{C} = \{a \in M \mid I(i, a) = 1,\ \forall i \in S\}.$$

Dually, the *extent* of any set of attributes $A \subseteq M$ is defined as the set of objects satisfying all attributes in A:

$$A'_\mathcal{C} = \{i \in N \mid I(i, a) = 1,\ \forall a \in A\}.$$

To avoid a heavy notation, we write simply S', A' for the intent of S and the extent of A, when the meaning is clear. Also, we write S'' or A'' instead of $(S')'$ and $(A')'$. Two fundamental properties of $'$ seen as a mapping from N to M or the converse is that $'$ is antimonotone, i.e., $S \subseteq T$ implies $S' \supseteq T'$ for every $S, T \in 2^N$ or 2^M, and *extensive*, i.e.

$$S'' \supseteq S, \quad A'' \supseteq A \quad \forall S \subseteq N, A \subseteq M. \tag{1}$$

A *concept* in $\mathcal{C}$ is a pair (S, A) with $S \subseteq N$ and $A \subseteq M$ such that $S = A'$ and $A = S'$. Equivalently, a concept is a maximal rectangle of "1" in the incidence matrix, or it is $(N, \emptyset)$ if $N' = \emptyset$, or $(\emptyset, M)$ if $M' = \emptyset$. Note that for any $S \subseteq N$ and $A \subseteq M$, the pairs (S'', S') and (A', A'') are concepts.

We denote by $L_{\mathcal{C}}$ the set of all concepts in $\mathcal{C}$, and endow it with a partial order $\leq$ defined by

$$(S, A) \leq (T, B) \text{ if } S \subseteq T$$

(equivalently, if $B \supseteq A$). Then $(L_{\mathcal{C}}, \leq)$ is a lattice, called the *concept lattice*, with supremum and infimum given by

$$\begin{aligned} (S, A) \wedge (T, B) &= ((S \cap T), (S \cap T)') \\ (S, A) \vee (T, B) &= ((A \cap B)', (A \cap B)). \end{aligned}$$

The top and bottom elements of this lattice are (N, N') and (M', M) respectively. It is important to note that any finite lattice is isomorphic to a concept lattice.

Given a context $\mathcal{C}$ and its concept lattice $L_{\mathcal{C}}$, the *lattice of extents* $(L^N_{\mathcal{C}}, \subseteq)$ is defined by the set

$$L^N_{\mathcal{C}} = \{S \subseteq N \mid (S, S') \in L_{\mathcal{C}}\}.$$

Similarly, we define the *lattice of intents* $(L^M_{\mathcal{C}}, \subseteq)$ as the set

$$L^M_{\mathcal{C}} = \{A \subseteq M \mid (A', A) \in L_{\mathcal{C}}\}.$$

Clearly, the lattices $L_{\mathcal{C}}, L^N_{\mathcal{C}}, (L^M_{\mathcal{C}})^\partial$ are isomorphic.

Example 1. Consider $N = \{1, 2, 3, 4\}$, $M = \{a, b, c\}$, and the incidence table given in Figure 1. The lattices $L_{\mathcal{C}}$, $L^N_{\mathcal{C}}$ and $L^M_{\mathcal{C}}$ are shown on the right of the table. For ease of notation, sets like $\{2, 4\}$ and $\{b, c\}$ are denoted by 24 and bc.

	a	b	c
1	×	×	
2		×	×
3	×		
4			×

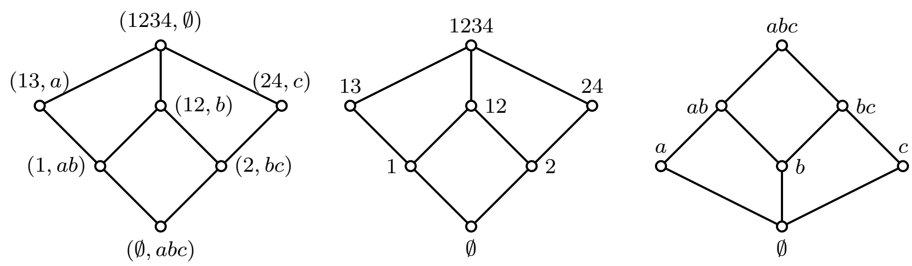


Fig. 1. From left to right: The incidence table of a context, its concept lattice, the lattices of extents, and the lattice of intents

2.3 Games on concept lattices

We consider a context $\mathcal{C} = (N, M, I)$, the lattice of concepts $L_{\mathcal{C}}$, the lattice of extents $L_{\mathcal{C}}^N$ and the lattice of intents $L_{\mathcal{C}}^M$. We assume that no attribute is superfluous, i.e., the top element of $L_{\mathcal{C}}$ is $(N, \emptyset)$ (no attribute is satisfied by all objects), however, the bottom element (M', M) may be with $M' \neq \emptyset$ (there are objects satisfying all attributes).

To each concept $(A, A') \in L_{\mathcal{C}}$, we assign a number $v(A, A') \in \mathbb{R}$ (its meaning could be benefit, cost, evaluation, certainty degree of occurrence, etc.). We call the pair $(\mathcal{C}, v)$ a *cooperative game on concepts* or *concept game* for short, and impose the restriction $v(\emptyset, M) = 0$ whenever $(\emptyset, M) \in L_{\mathcal{C}}$. We denote by CCG the set of all concept games.

We derive from v two mappings $v_N : L_{\mathcal{C}}^N \to \mathbb{R}$ and $v_M : L_{\mathcal{C}}^M \to \mathbb{R}$ defined by

$$v_N(S) = v(S, S') - v(M', M) \qquad (S \in L_{\mathcal{C}}^N)$$
$$v_M(A) = v(N, \emptyset) - v(A', A) \qquad (A \in L_{\mathcal{C}}^M).$$

Note that $v_N(M') = 0$ and $v_M(\emptyset) = 0$ holds, i.e., these set functions vanish at the bottom of their respective lattices, and could thus be considered as cooperative games. Also, if v is monotone nondecreasing, then so are v_N and v_M (because $A \subseteq B$ implies $A' \supseteq B'$) (and similarly for monotone nonincreasing). We call v_N and v_M the *game on extents* and the *game on intents*, respectively.

Example 2. An immediate application of the above framework in cooperative game theory is: N is the set of players, and M is the set of attributes of players. Attributes can be thought of as any kind of property, or simply, as membership cards of any association, club, party, etc., that the players may possess. Now a coalition is feasible/stable iff it corresponds to the extent of a concept. Indeed, suppose players in some coalition $S \subseteq N$ meet and compare their attributes. The set of common attributes shared by the members of S is S'. However, there may be other players sharing exactly the same attributes S', so that they could join S to form a stable coalition, in the sense that these are the only players sharing these attributes. This coalition is S'', and (S'', S') is a concept.

Example 3. Games on concepts can also model the interplay between sellers and markets. Consider $N = \{1, 2, 3, 4, 5, 6\}$ where agent 1 has the patent of a product in a market, and the remaining agents are sellers who want to sell the product in this market. The market is divided into three submarkets $M = \{\alpha, \beta, \gamma\}$, but there are restrictions on which agent can sell in which market: only sellers 4, 5, 6 can sell in market α, only 2, 4, 6 can sell in β, and only 3 and 5 can sell in γ. In addition, we suppose that agent 1 is not a seller. We define a context $\mathcal{C} = (N, M, I)$ to represent this situation with the relation I defined by $I(i, a) = 1$ if i cannot sell in submarket a. Figure 2 gives the incidence table and concept lattice[2] of $\mathcal{C}$. Let us define a concept game as follows. We make the assumption that 1) Profits obtained by a coalition of sellers depend on the submarkets where they can develop their activity, but 2) Sellers cannot ban others from these submarkets if the relation I makes them eligible. By a pair $(S, A) \in N \times M$, we represent a situation where S is a set of agents and A is a set of submarkets where they cannot sell. However, not every pair (S, A) is admissible. Indeed, the agents in S cannot sell in the submarkets of

[2] Note that this lattice is isomorphic to the one of Example 1, although the incidence tables are completely different.

	α	β	γ
1	$\times$	$\times$	$\times$
2	$\times$		$\times$
3	$\times$	$\times$	
4			$\times$
5		$\times$	
6			$\times$

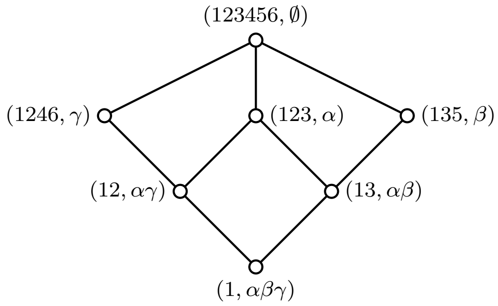


Fig. 2. The incidence table of $\mathcal{C}$ (left) and its concept lattice (right)

S', thus $A \supseteq S'$. Moreover, a seller has no interest to be in a situation which reduces his sale domain, thus $A \subseteq S'$, and therefore $A = S'$. By (1), this yields $A' = (S')' \supseteq S$. Now, 2) implies that $A' = S$, so finally (S, A) must be a context. We may take for v the following values (omitting braces and commas):

$$v(1, M) = 6, \quad v(13, \alpha\beta) = 16. \quad v(12, \alpha\gamma) = 20$$
$$v(123, \alpha) = 32, \quad v(135, \beta) = 28, \quad v(1246, \gamma) = 30, \quad v(N, \emptyset) = 60.$$

The value $v(1, M)$ represents the fixed payoff obtained by the patent owner if the product is sold, otherwise, $v(S, A)$ is the profit obtained by the sellers in S be selling outside the markets in A.

Let us compute the extent and intent games and see how to interpret them. We obtain:

S	1	12	13	123	135	1246	N
$v_N(S)$	0	14	10	26	22	24	54

A	γ	α	β	$\alpha\gamma$	$\alpha\beta$	$\alpha\beta\gamma$
$v_M(A)$	30	28	32	40	44	54

The value $v_N(S)$ represents the benefit realized by the coalition S, when the patent owner has been paid. Now, $v_M(A)$ represents the loss caused by not using the markets in A. It follows that the Shapley value (or any solution concept) would represent for v_N the contribution of each seller, and for v_M, it would represent in some sense the importance of each market, evaluated by a kind of average loss one faces when not using this market.

2.4 Set lattices and concept lattices

We investigate in this section the relation between concept lattices and set lattices on N, i.e., sublattices of $(2^N, \subseteq)$ (see, e.g., [17]). We begin by recalling some useful notions about finite lattices and posets. For x, y in a poset $(P, \preceq)$, $x \neq y$, we say that x *is covered by* y, or y *covers* x, denoted by $x \prec\cdot\, y$, if $x \preceq y$, and $x \preceq z \preceq y$ implies $x = z$ or $z = y$. For $x \preceq y$ in $(P, \preceq)$, a *maximal chain* from x to y is a sequence of elements $x = x_0, x_1, \ldots, x_p = y$ such that $x_0 \prec\cdot\, x_1 \prec\cdot\, \cdots \prec\cdot\, x_p$. Its *length* is p. The *height* of $x \in P$ is the length of a longest maximal chain from a minimal element to x. The *height* of P is the maximum over the height of all elements.

Given a poset $(P, \preceq)$, $x \in P$ is a *join-irreducible element* if it covers exactly one element. Dually, x is a *meet-irreducible element* if it is covered by exactly one element.

We denote respectively by $\mathcal{J}(P,\preceq)$ and $\mathcal{M}(P,\preceq)$ the sets of join-irreducible and meet-irreducible elements, which are subposets of $(P,\preceq)$. Whenever possible, we write $\mathcal{J}(P)$ and $\mathcal{M}(P)$ for simplicity. A subset $Q \subseteq P$ of a poset is a *downset* if $x \in Q$ and $y \in P$ such that $y \preceq x$ imply $y \in Q$. The set of all downsets of $(P,\preceq)$ is denoted by $\mathcal{O}(P,\preceq)$.

When $(P,\preceq)$ is a lattice, elements covering the bottom element are called *atoms*.

A lattice is *distributive* if $\vee,\wedge$ obey the algebraic distributivity law. A fundamental result due to Birkhoff says that a finite lattice $(L,\preceq)$ is isomorphic to $(\mathcal{O}(\mathcal{J}(L),\preceq),\subseteq)$ if and only if the lattice is distributive[3]. This means that a distributive lattice can be reconstructed from its join-irreducible elements by taking all downsets. The same statement with meet-irreducible elements holds as well, because L is distributive if and only if $(\mathcal{J}(L),\preceq)$ is isomorphic to $(\mathcal{M}(L),\preceq)$.

An application in game theory of the result of Birkhoff is the following: consider a set N of players endowed with a partial order $\preceq$. Then the set of downsets $\mathcal{O}(N,\preceq)$ forms a collection $\mathcal{F} \subseteq 2^N$ containing N and $\emptyset$, which is a distributive (set) lattice when ordered by inclusion, with supremum and infimum being union and intersection. Conversely, any collection $\mathcal{F} \subseteq 2^N$ of height n containing $N,\emptyset$ and closed under union and intersection arises that way (see Faigle and Kern [13]).

A *closure system on* N is a collection $\mathcal{F}$ of subsets of N which is closed under intersection and contains N, while a *dual closure system* is a collection closed under union and containing the empty set. Endowing a closure system (or a dual closure system) with inclusion order $\subseteq$, we obtain a poset with remarkable properties:

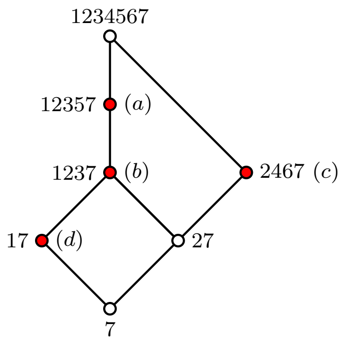
(i) Any lattice is isomorphic to a closure system, and to a dual closure system;
(ii) The lattice of extents of a context is a closure system, while the collection of complement sets of the lattice of intents, i.e., $\{A \in 2^M \mid A^c \in L^M_{\mathcal{C}}\}$, is a dual closure system. As a consequence, the lattices of extents and of intents are closed under intersection;
(iii) Conversely, any closure system $\mathcal{F}$ on N is the lattice of extents of some context. Specifically, the simplest context is $\mathcal{C}=(N,M,I)$, with M the set of meet-irreducible elements of $\mathcal{F}$, and $I(i,j)=1$ iff $i \in j$, with $i \in N$ and $j \in M$.

Example 4. Take $N=\{1,2,3,4,5,6,7\}$ and the closure system represented on Figure 3 (left). Its meet-irreducible elements are (in red): 12357, 1237, 2467, 17, which we denote a,b,c,d, respectively. The corresponding table is given in the middle, and the context lattice on the right of the figure.

We observe on the example that lines 4 and 6 are duplicate, line 7 is full, and line 3 is the intersection of lines 1 and 2. On the closure system, this corresponds respectively to the fact that 4 and 6 are always together in a concept, 7 is always present, and 3 is present whenever 1 and 2 are present. These situations are captured under the notion of macro-player and companion player (assuming N is the set of players).

Definition 1. *Let $\mathcal{F}$ be a closure system on N. A subset $K \subseteq N$, $|K|>1$, is a macro-player in $\mathcal{F}$ if either $K \subseteq S$ or $K \cap S = \emptyset$ for every nonempty $S \in \mathcal{F}$ (equivalently, no $S \in \mathcal{F}$ "separates" K, i.e., $S \cap K \neq \emptyset$ and $K \setminus S \neq \emptyset$).*

[3] We recall that $(\mathcal{J}(L),\preceq)$ is a subposet of $(L,\preceq)$. The downsets being subsets of $\mathcal{J}(L)$, they are ordered by inclusion. This should make the notation clear.



	a	b	c	d
1	×	×		×
2	×	×	×	
3	×	×		
4			×	
5	×			
6			×	
7	×	×	×	×

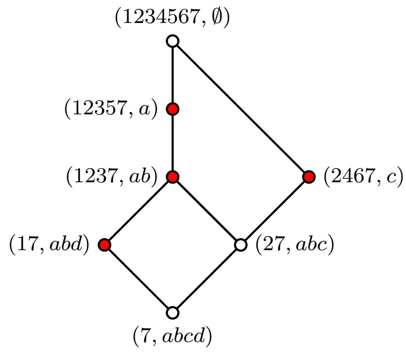


Fig. 3. From left to right: a closure system (meet-irreducible elements in red), the corresponding table and context lattice

Definition 2. *Let $\mathcal{F}$ be a closure system on N. A player $i \in N$ is a companion player of S, $S \subseteq N \setminus i$, if for all $T \in \mathcal{F}$,*

$$T \ni i \text{ if and only if } S \subseteq T.$$

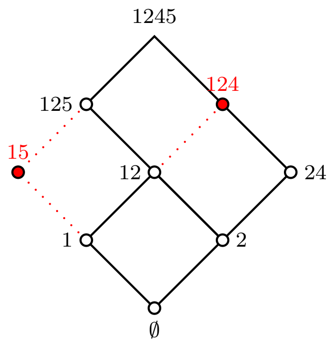
It is clear from the definition that macro-players arise as identical lines in the table, while a companion i of S corresponds to the situation where line i is the intersection of the lines in S. The following properties are noteworthy:

(i) If K, K' are maximal (w.r.t. inclusion) macro-players, then $K \cap K' = \emptyset$.
(ii) If M' (bottom of $\mathcal{F}$) is nonempty, then M' is a macro-player when $|M'| > 1$, and a companion player when $|M'| = 1$ (companion of $\emptyset$).
(iii) When $M' = \emptyset$, atoms which are not singletons are macro-players, but the converse is false. More precisely, a macro-player K is an atom if and only if $K \in \mathcal{F}$.
(iv) i is a companion of $\{j\}$ if and only if $\{i, j\}$ is a macro-player.
(v) If i and j are companion of the same S, then $\{i, j\}$ is a macro-player.

Consider a closure system $\mathcal{F}$ on N with $|N| = n$, and consider $\mathcal{J}(\mathcal{F})$ the set of its join-irreducible elements. The following is important to note.

(i) Suppose $|\mathcal{J}(\mathcal{F})| < n$. Then there exist either companion players or macro-players. Indeed, each join-irreducible element corresponds to a line in the incidence table, and each additional line must not create a new maximal rectangle with new attributes. We have
$$n = |\mathcal{J}(L)| + \sum_{i=1}^{p} |K_i| - p + c$$
where $K_1, \ldots, K_p$ are the maximal macro-players and c is the number of companion players which do not belong to some macro-player (see Ex. 4).
(ii) Suppose $|\mathcal{J}(\mathcal{F})| = n$. In this case, $\mathcal{F}$ is an *irreducible closure system* since there is no redundant line in the incidence table. Moreover, $\mathcal{F} \subseteq \mathcal{O}(\mathcal{J}(\mathcal{F}))$ where "missing" sets (i.e., those not in $\mathcal{O}(\mathcal{J}(\mathcal{F}))$) are necessarily unions of sets in $\mathcal{F}$, since taking the closure under union of $\mathcal{F}$ would give $\mathcal{O}(\mathcal{J}(\mathcal{F}))$.

Example 5 (Example 4 continued). Let us make the closure system of Example 4 irreducible by suppressing the superfluous elements 3, 6 and 7, so as to have $n = |\mathcal{J}(\mathcal{F})| = 4$. This gives the closure system represented on Figure 4 (solid lines), to which we give a slightly different shape, in order to make it apparent as a sublattice of $\mathcal{O}(\mathcal{J}(\mathcal{F}))$ (additional links in red dotted lines). The two missing sets are 15 and 124 (in red).



	a	b	c	d
1	×	×		×
2	×	×	×	
4			×	
5	×			

Fig. 4. Irredundant version of Figure 3

Remark 1. Evidently, the same observations can be made for attributes: one can define in the same way macro-attributes and companion attributes.

Let L be any lattice, with $N = \{1, 2, \dots\}$ the set of its join-irreducible elements, and $M = \{a, b, c, \dots\}$ the set of its meet-irreducible elements. The *irreducible closure system associated to* L is the set lattice $C(L)$ on N defined by

$$C(L) = \{J(x) \mid x \in L\}, \text{with } J(x) = \{i \in N \mid i \leq x\}.$$

Its bottom element is $\emptyset$. The *irreducible dual closure system associated to* L is the set lattice $O(L)$ on M defined by

$$O(L) = \{M(x) \mid x \in L\}, \text{with } M(x) = \{j \in M \mid j \ngeq x\}.$$

Its top element is M. The *irreducible concept lattice* associated to L is given by the *irreducible* context $\mathcal{C} = (N, M, I)$ with $I(i, j) = 1$ iff $i \leq j$. Then

$$L^N_{\mathcal{C}} = C(L) \text{ and } L^M_{\mathcal{C}} = \{A \subseteq M \mid A^c \in O(L)\}.$$

Example 6. Take the lattice on Figure 5 (left), its join-irreducible elements are in red, the meet-irreducible elements are in blue. The irreducible closure and dual closure systems (ordered by $\subseteq$) are depicted in the middle and on the right of the figure. By comparing with Example 1, one can see the above identity.

We say that two contexts $\mathcal{C}, \mathcal{C}'$ are *equivalent* if their concept lattices $L_{\mathcal{C}}, L_{\mathcal{C}'}$ are isomorphic. Based on the above facts, we know that $\mathcal{C}, \mathcal{C}'$ are equivalent to the same irreducible context, and they only differ by the adjunction of macro-players, companion players, and macro-attributes and companion attributes.

3 The Shapley value

Given a game $(\mathcal{C}, v)$ on a context, we define the extent Shapley value and the intent Shapley value as the Shapley value for the games on extents and on intents respectively. We begin with the extent Shapley value.

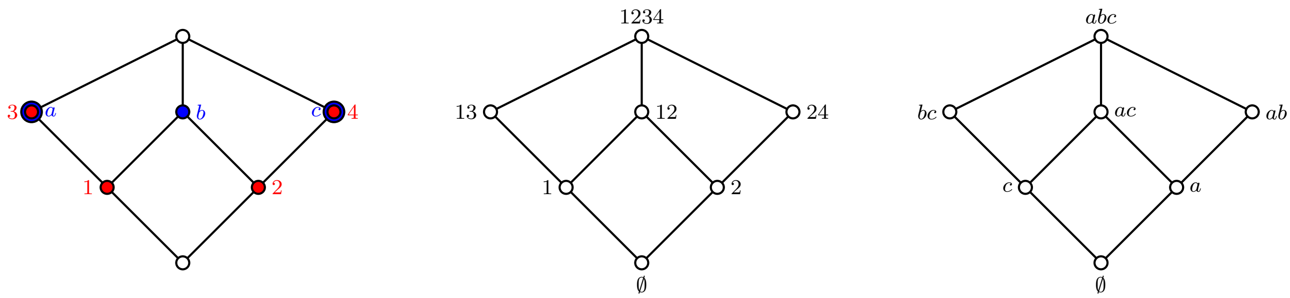


Fig. 5. From left to right: a lattice (in red: join-irreducible elements, in blue: meet-irreducible elements), and the corresponding irreducible closure and dual closure systems

3.1 The extent Shapley value

We consider the lattice of extents $L^N_{\mathcal{C}}$ of a context $\mathcal{C}$ and the game v_N defined on it.

Consider the set $CH(\mathcal{C})$ of all maximal chains from the bottom M' to the top N in $L^N_{\mathcal{C}}$ (equivalently, in $L_{\mathcal{C}}$), and denote its cardinality by $ch(\mathcal{C})$. Consider a given maximal chain $C \in CH(\mathcal{C})$, letting $C = M' = S_0 \subset S_1 \cdots \subset S_k = N$, and a player i. Denote by T^i_C and S^i_C respectively, the last set in the sequence which does not contain i, and the first set containing i.

The *extent Shapley value* of $(\mathcal{C}, v)$, denoted by $\Phi^{ex}(\mathcal{C}, v)$, is defined to be the Shapley value $\Phi(v_N)$ of the game on extents, given by

$$\Phi^{ex}_i(\mathcal{C}, v) = \Phi_i(v_N) = \begin{cases} \dfrac{1}{ch(\mathcal{C})} \displaystyle\sum_{C \in CH(\mathcal{C})} \dfrac{1}{|S^i_C \setminus T^i_C|} \big(v_N(S^i_C) - v_N(T^i_C)\big), & \text{if } i \notin M' \\ \dfrac{v_N(M')}{|M'|}, & \text{otherwise.} \end{cases} \tag{2}$$

This definition is a natural generalization of the values introduced by Faigle and Kern [13], and Bilbao and Edelman [5].

We formulate properties to axiomatize the extent Shapley value. Let F be any value over the set of concept games.

Taking into account that we consider $v(M', M)$ as a separable payoff to players in M', we propose:

Separable payoff axiom (SP): If $(\mathcal{C}, v) \in CCG$ and $\mathcal{C} = (N, M, I)$ then one has

$$\sum_{i \in M'} F_i(\mathcal{C}, v) = v(M', M).$$

As for the classical Shapley value, we look for efficient payoff vectors.

Efficiency axiom (E): For all $(\mathcal{C}, v) \in CCG$, one has

$$\sum_{i \in N} F_i(\mathcal{C}, v) = v(N, \emptyset).$$

All the agents in a macro-player are equivalent for the concept lattice, thus their worths should be the same.

Macro-player axiom (MP): If $(\mathcal{C}, v) \in CCG$ with $\mathcal{C} = (N, M, I)$ and K is a macro-player in $L^N_{\mathcal{C}}$, then

$$F_i(\mathcal{C}, v) = F_j(\mathcal{C}, v) \quad \forall i, j \in K.$$

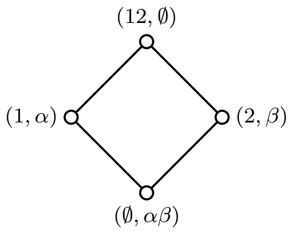
A context $\mathcal{C}_2 = (N_2, M_2, I_2)$ is *concatenable* to a context $\mathcal{C}_1 = (N_1, M_1, I_1)$ if $N_1 = (M_2)'_{\mathcal{C}_2}$ and $M_1 \cap M_2 = \emptyset$. The result of the concatenation of the two concatenable contexts is a new context $\mathcal{C}_2 * \mathcal{C}_1 = (N, M, I)$, where $N = N_2$, $M = M_1 \cup M_2$ and

$$I(i, a) = \begin{cases} I_1(i, a), & \text{if } i \in N_1, a \in M_1 \\ I_2(i, a), & \text{if } i \in N_2 \setminus N_1, a \in M_2 \\ 1, & \text{if } i \in N_1, a \in M_2 \\ 0, & \text{if } i \in N_2 \setminus N_1, a \in M_1. \end{cases}$$

As is easy to see, concatenation amounts to the concatenation of the two incidence tables and hence to the concatenation of the two concept lattices $\{(S, A \cup M_2) \mid (S, A) \in L_{\mathcal{C}_1}\}$ and $L_{\mathcal{C}_2}$.

Example 7. Consider $N_1 = \{1, 2\}$, $N_2 = \{1, 2, 3, 4, 5\}$, $M_1 = \{\alpha, \beta\}$ and $M_2 = \{a, b, c\}$. The two incidence tables and the concept lattices $L_{\mathcal{C}_1}, L_{\mathcal{C}_2}$ are given on Figure 6. The

	α	β
1	×	
2		×



	a	b	c
1	×	×	×
2	×	×	×
3	×		
4		×	
5			×

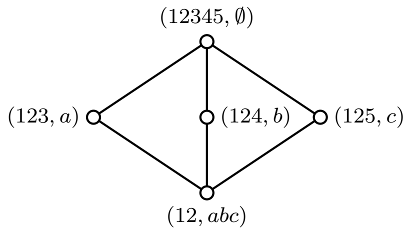


Fig. 6. Two contexts $\mathcal{C}_1, \mathcal{C}_2$ represented by their table and concept lattice

result of the concatenation $\mathcal{C}_2 * \mathcal{C}_1$ is shown on Figure 7.

	a	b	c	α	β
1	×	×	×	×	
2	×	×	×		×
3	×				
4		×			
5			×		

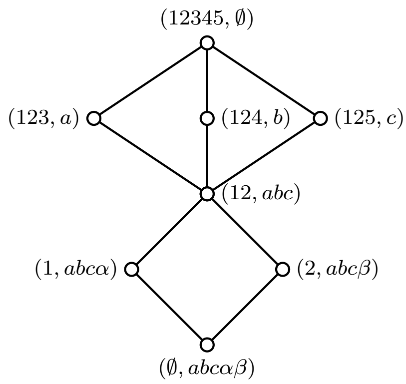


Fig. 7. The concatenation $\mathcal{C}_2 * \mathcal{C}_1$ of the two contexts of Figure 6

The concatenation of contexts should not change the payoffs of the players.

Concatenation axiom (C): For all $(\mathcal{C}_1, v_1)$, $(\mathcal{C}_2, v_2) \in CCG$ such that $\mathcal{C}_2$ is concatenable to $\mathcal{C}_1$, and $v_1(N_1, \emptyset) = v_2((M_2)'_{\mathcal{C}_2}, M_2)$, one has

$$F_i(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \begin{cases} F_i(\mathcal{C}_2, v_2), & \text{if } i \in N_2 \setminus N_1 \\ F_i(\mathcal{C}_1, v_1), & \text{if } i \in N_1, \end{cases}$$

where $v_2 * v_1$ is a concept game on $\mathcal{C} = \mathcal{C}_2 * \mathcal{C}_1$, defined by

$$(v_2 * v_1)(S, A) = \begin{cases} v_1(S, A \setminus M_2), & \text{if } (A \setminus M_2)'_{\mathcal{C}} \subseteq N_1 \\ v_2(S, A), & \text{if } N_1 \subseteq S. \end{cases}$$

Our last axiom is related to the decomposition of the concept lattice by restricting on some special attributes. We need some additional notions. Let $\mathcal{C} = (N, M, I)$ be a context, and $a \in M$ be an attribute. The *restriction of* $\mathcal{C}$ *to* a is the context $\mathcal{C}_{|a} = (N, M, I_{|a})$ defined by

$$I_{|a}(i, b) = \begin{cases} I(i, b), & \text{if } i \in a'_{\mathcal{C}} \\ 0, & \text{otherwise} \end{cases}$$

for each $i \in N$ and $b \in M$. Observe that $L_{\mathcal{C}_{|a}}$ is a sublattice of $L_{\mathcal{C}}$, given by

$$L_{\mathcal{C}_{|a}} = \{(S, A) \in L_{\mathcal{C}} \mid a \in A\} \cup \{(N, \emptyset)\}. \tag{3}$$

Indeed, if $(S, A) \in L_{\mathcal{C}_{|a}}$ with $S \neq N$, we have $I_{|a}(i, b) = 1$ for all $i \in S$ and $b \in A$. Hence $S \subseteq a'_{\mathcal{C}}$, $a \in A$ and $A = S'_{\mathcal{C}_{|a}} = S'_{\mathcal{C}}$.

Let $\mathcal{C} = (N, M, I)$ be a context. An attribute $a \in M$ is *superfluous* if $a'_{\mathcal{C}} = N$. An attribute is *separating* if it is not superfluous, and there is no nonsuperfluous $b \in M$, $b \neq a$, such that $b' \supset a'$. We denote by $\mathrm{Sep}(\mathcal{C})$ the set of separating attributes of $\mathcal{C}$.

A characteristic property of separating attributes is the following.

Lemma 1. *$a \in M$ is a separating attribute if and only if $(a', a) \in L_{\mathcal{C}}$ or a belongs to some macro-attribute A and $(A', A) \in L_{\mathcal{C}}$.*

Proof. Suppose that a does not belong to a macro-attribute (i.e., there is no $b \in M$, $b \neq a$, such that $b' = a'$). We have to prove that a separating is equivalent to $a'' = a$. By (1), $a'' \neq a$ is equivalent to $a'' \supset a$, i.e., there exists $b \in M$, $b \neq a$, such that $b \in a''$. This is equivalent to $b' \supset a'$, which means that a is not separating.

Now, if a belongs to a macro-attribute, the macro-attribute plays the role of a. □

It follows from (3), Lemma 1 and the fact that $(N, \emptyset) \in L_{\mathcal{C}_{|a}}$ that any maximal chain in $L_{\mathcal{C}}$ from (M', M) to $(N, \emptyset)$ is a maximal chain in $L_{\mathcal{C}_{|a}}$ for some separating attribute a, and vice versa. In other words, one can make a partition of the lattice $L_{\mathcal{C}}$ into maximal chains, according to separating attributes or separating macro-attributes. The following example illustrates this.

Example 8. We consider 5 players and 5 attributes, and the context defined by the following table:

	a	b	c	d	e
1	×	×	×		
2		×	×		
3			×		×
4		×		×	×
5			×	×	×

The separating attributes are b, c and e. We show on Figure 8 the concept lattice and its decomposition according to b, c, e.

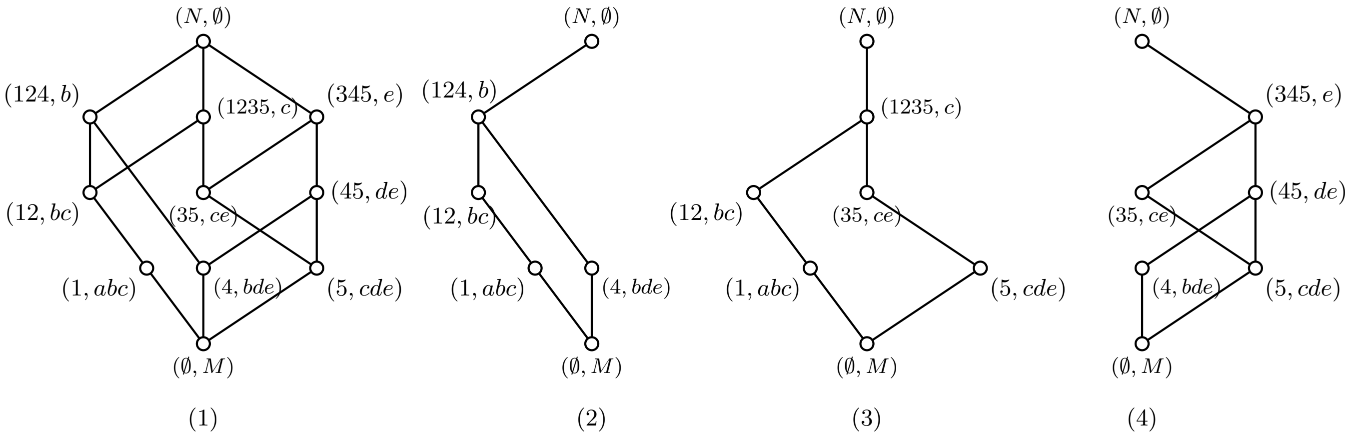


Fig. 8. (1) The concept lattice $L_{\mathcal{C}}$ and its restrictions to b (2), c (3) and to e (4)

The following decomposition axiom is similar to the hierarchical strength in Faigle and Kern [13].

Decomposition axiom (D): Let $(\mathcal{C}, v)$ be a concept game. It holds

$$ch(\mathcal{C})F(\mathcal{C}, v) = \sum_{a \in \mathrm{Sep}(\mathcal{C})} ch(\mathcal{C}_{|a})F(\mathcal{C}_{|a}, v^a),$$

where v^a is the restriction of v to the concepts in $\mathcal{C}_{|a}$ for each $a \in \mathrm{Sep}(\mathcal{C})$.

We show now that the extent Shapley value satisfies all these axioms.

Theorem 1. *The extent Shapley value satisfies (SP), (E), (MP), (C) and (D).*

Proof. – Separable payoff: it is obviously satisfied.

– Efficiency: Let $(\mathcal{C}, v) \in CCG$ with $\mathcal{C} = (N, M, I)$. We have, using (2),

$$\begin{aligned}\sum_{i \in N} \Phi_i^{ex}(\mathcal{C}, v) &= \sum_{i \in M'} \Phi_i^{ex}(\mathcal{C}, v) + \sum_{i \in N \setminus M'} \Phi_i^{ex}(\mathcal{C}, v) \\ &= v(M', M) + \frac{1}{ch(\mathcal{C})} \sum_{C \in CH(\mathcal{C})} (v(N, \emptyset) - v(M', M)) = v(N, \emptyset).\end{aligned}$$

– Macro-players: Let $(\mathcal{C}, v) \in CCG$ and K be a macro-player in $L_{\mathcal{C}}^N$. If $K = M'$, by definition, every two players in K receive the same payoff with Φ^{ex}. Suppose now $K \neq M'$, i.e., $K \subseteq N \setminus M'$. In that case, for any $i, j \in K$ and for any $C \in CH(\mathcal{C})$, $S_C^i = S_C^j$ and $T_C^i = T_C^j$. Thus $\Phi_i^{ex}(\mathcal{C}, v) = \Phi_j^{ex}(\mathcal{C}, v)$ for every $i, j \in K$.

– Concatenation: We consider two concatenable contexts $\mathcal{C}_1 = (N_1, M_1, I_1)$, $\mathcal{C}_2 = (N_2, M_2, I_2)$ with $N_1 = (M_2)'_{\mathcal{C}_2}$. Observe that $ch(\mathcal{C}_2 * \mathcal{C}_1) = ch(\mathcal{C}_2)\, ch(\mathcal{C}_1)$. If $i \in (M_1)'_{\mathcal{C}_1}$ then

$$\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \frac{(v_2 * v_1)\left((M_1)'_{\mathcal{C}_1}, M_1 \cup M_2\right)}{|(M_1)'_{\mathcal{C}_1}|} = \Phi_i^{ex}(\mathcal{C}_1, v_1).$$

If $i \in N_1 \setminus (M_1)'_{\mathcal{C}_1}$, then for any maximal chain C in $\mathcal{C}_2 * \mathcal{C}_1$ the transition T_C^i to S_C^i occurs $ch(\mathcal{C}_2)$ times as the same transition in C restricted to $\mathcal{C}_1$. Hence, we have

$$\begin{aligned}
&\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \\
&= \frac{1}{ch(\mathcal{C}_2 * \mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_2 * \mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[(v_1 * v_2)\left(S_C^i, (S_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right) - (v_1 * v_2)\left(T_C^i, (T_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right)\right] \\
&= \frac{1}{ch(\mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[v_1\left(S_C^i, (S_C^i)'_{\mathcal{C}_1}\right) - v_1\left(T_C^i, (T_C^i)'_{\mathcal{C}_1}\right)\right] = \Phi_i^{ex}(\mathcal{C}_1, v_1).
\end{aligned}$$

If $i \in N_2 \setminus N_1$, then for each maximal chain C in $\mathcal{C}_2 * \mathcal{C}_1$ the transition T_C^i to S_C^i occurs $ch(\mathcal{C}_1)$ times as the same transition in C restricted to $\mathcal{C}_2$. Hence

$$\begin{aligned}
&\Phi_i^{ex}(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \\
&= \frac{1}{ch(\mathcal{C}_2 * \mathcal{C}_1)} \sum_{C \in CH(\mathcal{C}_2 * \mathcal{C}_1)} \frac{1}{|S_C^i \setminus T_C^i|} \left[(v_1 * v_2)\left(S_C^i, (S_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right) - (v_1 * v_2)\left(T_C^i, (T_C^i)'_{\mathcal{C}_2 * \mathcal{C}_1}\right)\right] \\
&= \frac{1}{ch(\mathcal{C}_2)} \sum_{C \in CH(\mathcal{C}_2)} \frac{1}{|S_C^i \setminus T_C^i|} \left[v_2\left(S_C^i, (S_C^i)'_{\mathcal{C}_2}\right) - v_2\left(T_C^i, (T_C^i)'_{\mathcal{C}_2}\right)\right] = \Phi_i^{ex}(\mathcal{C}_2, v_2).
\end{aligned}$$

– Decomposition: consider the decomposition of $\mathcal{C}$ into $\mathcal{C}_{|a}$, $a \in \mathrm{Sep}(\mathcal{C})$, according to the separating attributes. By the definition of v^a, $a \in \mathrm{Sep}(\mathcal{C})$, we have for every player i

$$\begin{aligned}
\Phi_i^{ex}(\mathcal{C}, v) &= \frac{1}{ch(\mathcal{C})} \sum_{C \in CH(\mathcal{C})} \frac{1}{|S_C^i \setminus T_C^i|} \left[v_N\left(S_C^i\right) - v_N\left(T_C^i\right)\right] \\
&= \frac{1}{ch(\mathcal{C})} \sum_{a \in \mathrm{Sep}(\mathcal{C})} \sum_{C \in CH(\mathcal{C}_{|a})} \frac{1}{|S_C^i \setminus T_C^i|} \left[v^a\left(S_C^i, (S_C^i)'_{\mathcal{C}_{|a}}\right) - v^a\left(T_C^i, (T_C^i)'_{\mathcal{C}_{|a}}\right)\right] \\
&= \frac{1}{ch(\mathcal{C})} \sum_{a \in \mathrm{Sep}(\mathcal{C})} ch(\mathcal{C}_{|a}) \Phi_i^{ex}\left(\mathcal{C}_{|a}, v^a\right).
\end{aligned}$$

□

Finally, we prove that the extent Shapley value is the only value for concept games that satisfies the above axioms.

Theorem 2. *The extent Shapley value is the only value over concept games satisfying (SP), (E), (MP), (C) and (D).*

Proof. We have already proved in Theorem 1 that the extent Shapley value satisfies all these axioms. It remains to show that they uniquely determine the value.

Let $(\mathcal{C}, v) \in CCG$ be a concept game with $\mathcal{C} = (N, M, I)$ with $M = \{a\}$, or more generally $M = A$, where A is a macro-attribute (simple context). We have in that case $M' = \{i \in N \mid I(i, a) = 1\}$ for any $a \in A$ (recall that all columns in I are identical for $a \in A$), so that the concept lattice is reduced to $\{(M', A), (N, \emptyset)\}$. Observe that whenever $|M'| > 1$, M' is a macro-player, and similarly $N \setminus M'$ is a macro-player as well if $|N \setminus M'| > 1$. Suppose first that $M' = \emptyset$. The macro-player axiom imposes that $F_i(\mathcal{C}, v) = F_j(\mathcal{C}, v)$ for all $i, j \in N$, hence by the efficiency axiom, it follows that

$$F_i(\mathcal{C}, v) = \frac{1}{|N|} v(N, \emptyset), \quad \forall i \in N,$$

so that F is uniquely determined for that type of game. Suppose now that M' is reduced to a singleton, say $\{i\}$. The separable payoff axiom imposes that $F_i(\mathcal{C}, v) = v(M', M)$. If $|M'| > 1$, M' is a macro-player, and by the macro-player axiom, it follows that for every $i, j \in M'$, $F_i(\mathcal{C}, v) = F_j(\mathcal{C}, v)$. Now, the separable payoff axiom implies

$$\sum_{j \in M'} F_j(\mathcal{C}, v) = v(M', M),$$

so that finally $F_i(\mathcal{C}, v) = \dfrac{v(M', M)}{|M'|}$, for all $i \in M'$. We can proceed similarly with the remaining players in $N \setminus M'$: applying (MP) and (E) finally yields

$$F_i(\mathcal{C}, v) = \frac{1}{|N \setminus M'|} \big(v(N, \emptyset) - v(M', M) \big), \quad \forall i \in N \setminus M'.$$

As a conclusion, F is uniquely determined for any game $(\mathcal{C}, v)$ with $\mathcal{C} = (N, A, I)$, where A is a macro-attribute (in particular a singleton $\{a\}$).

Consider now any concept game $(\mathcal{C}, v)$ and the decomposition of $\mathcal{C}$ into $\mathcal{C}_{|a}$, $a \in \mathrm{Sep}(\mathcal{C})$. The decomposition axiom implies that if F is uniquely determined on each $\mathcal{C}_{|a}$, then F is uniquely determined on $\mathcal{C}$. Hence we consider now $\mathcal{C}_{|a}$ for some $a \in \mathrm{Sep}(\mathcal{C})$. By Lemma 1 and (3), for any $(S, A) \in L_{\mathcal{C}_{|a}}$, we have $a \in A$ and $S \subseteq a'$. It follows from the definition of concatenable contexts that, when a does not belong to a macro-attribute, $L_{\mathcal{C}_{|a}}$ can be written as the concatenation of the context $\mathcal{C}^a = (N, \{a\}, I^a)$ whose concept lattice is simply $\{(a', a), (N, \emptyset)\}$ and the context $\mathcal{C}^{-a} = (a', M \setminus \{a\}, I^{-a})$, with I^{-a} obtained from $I_{|a}$ by deleting attribute a and players in $N \setminus a'$:

$$\mathcal{C}_{|a} = \mathcal{C}^a * \mathcal{C}^{-a}$$

(if a belongs to some macro-attribute A, replace a by A). Observe that $\mathcal{C}^a$ is a simple context. Repeating the same process on $\mathcal{C}^{-a}$, we arrive in a finite number of steps at

$$\mathcal{C}_{|a} = \mathcal{C}^a * (\mathcal{C}^{-a})^b * ((\mathcal{C}^{-a})^{-b})^c * \cdots$$

where b is a separating attribute of $\mathcal{C}^{-a}$, etc., and all contexts in the above formula are simple. By a repeated application of the concatenation axiom, we ultimately determine F on $\mathcal{C}_{|a}$, for every $a \in \mathrm{Sep}(\mathcal{C})$. $\square$

3.2 The intent Shapley value

We proceed in a similar way as for the extent value. We consider the lattice of intents $L^M_{\mathcal{C}}$ of a context $\mathcal{C}$ and the game v_M defined on it. We note that the set of maximal chains $CH(\mathcal{C})$ is isomorphic to the set of chains in $L^M_{\mathcal{C}}$. For a given maximal chain $C = \emptyset \subset S_1 \subset \cdots \subset S_k = M$ in $L^M_{\mathcal{C}}$ and an attribute $a \in M$, let B^a_C and A^a_C be respectively the last set in the sequence which does not contain a, and the first set containing a.

The *intent Shapley value* of $(\mathcal{C}, v)$, denoted by $\Phi^{in}(\mathcal{C}, v)$, is defined to be the Shapley value $\Phi(v_M)$ of the game of intents, given by

$$\Phi^{in}_a(\mathcal{C}, v) = \Phi_a(v_M) = \frac{1}{ch(\mathcal{C})} \sum_{C \in CH(\mathcal{C})} \frac{1}{|A^a_C \setminus B^a_C|}(v_M(A^a_C) - v_M(B^a_C)).$$

We formulate several properties. Let Ψ be any value over the set of concept games.

Efficiency axiom (E): $\sum_{a \in M} \Psi_a(\mathcal{C}, v) = v_M(M) = v(N, \emptyset) - v(M', M)$, for all $(\mathcal{C}, v) \in CCG$.

Let $K \subseteq M$, $|K| > 1$. We say that the set K is a *macro-attribute* if for any $S' \in L^M_{\mathcal{C}}$, $S' \neq \emptyset$, we have $K \subseteq S'$ or $K \cap S' = \emptyset$.

Macro-attribute axiom (MA): If $(\mathcal{C}, v) \in CCG$ and K is a macro-attribute in $L^M_{\mathcal{C}}$, then

$$\Psi_a(\mathcal{C}, v) = \Psi_b(\mathcal{C}, v) \qquad \forall a, b \in K.$$

Using the definition of the concatenation of contexts as given for the case of extent games, we introduce the following axiom.

Concatenation axiom (C): For all $(\mathcal{C}_1, v_1), (\mathcal{C}_2, v_2) \in CCG$ such that $\mathcal{C}_2$ is concatenable to $\mathcal{C}_1$, and $v_1(N_1, \emptyset) = v_2((M_2)'_{\mathcal{C}_2}, M_2)$, it holds

$$\Psi_a(\mathcal{C}_2 * \mathcal{C}_1, v_2 * v_1) = \begin{cases} \Psi_a(\mathcal{C}_2, v_2), & \text{if } a \in M_2 \\ \Psi_a(\mathcal{C}_1, v_1), & \text{if } a \in M_1 \setminus M_2 \end{cases}$$

where $v_2 * v_1$ is defined as for the extent value.

We consider a concept $\mathcal{C} = (N, M, I)$ and its decomposition according to separating players (defined similarly as separating attributes).

Decomposition axiom (D): Let $(\mathcal{C}, v)$ be a concept game. It holds

$$ch(\mathcal{C})\Psi(\mathcal{C}, v) = \sum_{i \in \mathrm{Sep}'(\mathcal{C})} ch(\mathcal{C}_{|i}, v^i)\Psi(\mathcal{C}_{|i}, v^i),$$

where $\mathrm{Sep}'(\mathcal{C})$ is the set of separating players, $\mathcal{C}_{|i}$ the restriction of $\mathcal{C}$ to player i, and v^i the restriction of v to $\mathcal{C}_{|i}$.

Theorem 3. *The intent Shapley value is the unique value over the set of concept games which satisfies (E), (MA), (C) and (D).*

Proof is similar to the case of the extent Shapley value and is therefore omitted.

4 The core

Given a game $(\mathcal{C}, v)$ on a context, we consider the cores of the games on extents and intents:

$$\begin{aligned}\mathrm{core}(v_N) &= \{x \in \mathbb{R}^N \mid x(S) \geq v_N(S), S \in L^N_{\mathcal{C}} \text{ and } x(N) = v_N(N)\}\\ \mathrm{core}^*(v_M) &= \{y \in \mathbb{R}^M \mid y(A) \leq v_M(A), A \in L^M_{\mathcal{C}} \text{ and } y(M) = v_M(M)\}.\end{aligned}$$

(Note: $\mathrm{core}^*(v_M)$ is the anti-core, i.e., the set of vectors y such that $-y$ is in the core of $-v_M$). Let us call them for convenience the extent core and the intent core respectively.

We can write the intent core in a more convenient way. For any vector $y \in \mathrm{core}^*(v_M)$, we have

$$\begin{aligned}&y(A) \leq v_M(A) = v(N, \emptyset) - v(A', A), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v_M(M) = v(N, \emptyset) - v(M', M)\\ &\Leftrightarrow y(M) - y(M \setminus A) \leq v(N, \emptyset) - v(A', A), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v(N, \emptyset) - v(M', M)\\ &\Leftrightarrow y(M \setminus A) \geq v(A', A) - v(M', M), \forall A \in L^M_{\mathcal{C}}, \text{ and } y(M) = v_M(M) = v(N, \emptyset) - v(M', M)\end{aligned}$$

i.e., $y \in \mathrm{core}(\overline{v}_M)$, with $\overline{v}_M(A) = v((M \setminus A)', M \setminus A) - v(M', M)$, for all $A \in \overline{L^M_{\mathcal{C}}}$, where $\overline{L^M_{\mathcal{C}}} = \{A \subseteq M \mid M \setminus A \in L^M_{\mathcal{C}}\}$ is the dual closure system associated to $L^M_{\mathcal{C}}$. This proves $\mathrm{core}^*(v_M) = \mathrm{core}(\overline{v}_M)$. Note that if $M' = \emptyset$, $\overline{v}$ coincides with the conjugate of v_M, that is, $\overline{v}_M(A) = v_M(M) - v_M(A^c)$.

Example 9 (Example 1 continued). Let us define the following game on the concept lattice of Figure 1:

(S, S')	$(1, ab)$	$(2, bc)$	$(13, a)$	$(12, b)$	$(24, c)$	$(1234, \emptyset)$
$v(S, S')$	10	20	50	40	40	100

We obtain

$$\mathrm{core}(v_N) = \begin{cases} x_1 & \geq 10\\ x_2 & \geq 20\\ x_1 + x_2 & \geq 40\\ x_1 + x_3 & \geq 50\\ x_2 + x_4 & \geq 40\\ x_1 + x_2 + x_3 + x_4 & = 100\end{cases}, \quad \mathrm{core}^*(v_M) = \begin{cases} y_c & \geq 10\\ y_a & \geq 20\\ y_b + y_c & \geq 50\\ y_a + y_c & \geq 40\\ y_a + y_b & \geq 40\\ y_a + y_b + y_c & = 100\end{cases}$$

which are not empty since $x = (20, 20, 30, 30) \in \mathrm{core}(v_N)$ and $y = (30, 30, 40) \in \mathrm{core}^*(v_M)$.

An important consequence of the above facts, we find that the study of the extent and intent cores amounts to the study of the core of games on closure systems (closed under intersection) and on dual closure systems (closed under union). In what follows we study in depth the structure of the extent core, especially its conic part. Results on the intent core will be obtained by duality. In the whole section, $\mathcal{F}$ denotes any collection of sets.

4.1 Nonemptiness

We ask when the cores are nonempty. As said in Section 2.1, the core of a game on a subcollection $\mathcal{F}$ of 2^N is nonempty if and only if the game is balanced in the usual sense. Hence, $\mathrm{core}(v_N)$ is nonempty if and only if v_N is balanced, and $\mathrm{core}^*(v_M)$ is nonempty if and only if $\overline{v}_M$ is balanced.

The case $M' \neq \emptyset$ deserves some attention, because then $\mathrm{core}(v_N)$ is never empty. Indeed, it is not difficult to see that the only balanced collection in $L^N_{\mathcal{C}}$ is $\{N\}$, whence any game on the lattice of extents is balanced. There is no such conclusion for $\mathrm{core}^*(v_M)$ because $N' = \emptyset$.

To avoid triviality, we suppose in the rest of this section that $M' = \emptyset$. There are two natural question coming up to mind:

(i) Is nonemptiness of the extent core or intent core invariant to the chosen context, among all equivalent ones?

(ii) Is there any relation between nonemptiness of the extent and intent cores?

Concerning the first question, we know by Section 2.4 that given a lattice L, we generate the corresponding irreducible context (N, M, I) by taking N, M to be the set of its join- and meet-irreducible elements. Also, all other equivalent contexts are obtained by duplicating lines or columns (creating macro-players and macro-attributes), or by adding lines or columns that are intersection of others (creating companion players or attributes), or by adding a full line or column, but the latter is discarded as this produces $M', N' \neq \emptyset$. We show that these additions do not change the nonemptiness conditions for both cores.

Theorem 4. *Let L be a lattice, and $\mathcal{C} = (N, M, I)$ be the corresponding irreducible context. Then for any game v on $\mathcal{C}$, nonemptiness of $\mathrm{core}(v_N), \mathrm{core}^*(v_M)$ implies nonemptiness of $\mathrm{core}(v_{N'}), \mathrm{core}^*(v_{M'})$ for any equivalent context (N', M', I').*

Proof. The case of macro-players or attributes is clear: it is just a matter of change of variable. If $K = \{i_1, \dots, i_k\}$ is a macro-player, define $x_K = x_{i_1} + \cdots + x_{i_k}$ and make the substitution in the system of inequalities defining the core. The conditions for nonemptiness are not changed.

It remains to examine the case of companion players. We deal with the extent core first. We take $\mathcal{C} = (N, M, I)$ irreducible, put for convenience $\mathcal{F} = L^N_{\mathcal{C}}$, and consider $N^* = N \cup \{i\}$, where i is a companion player of $\emptyset \neq K \subseteq N$. The new context is denoted by $\mathcal{C}^* = (N^*, M, I^*)$, and we set $\mathcal{F}^* = L^{N^*}_{\mathcal{C}^*}$, where $\mathcal{F}^* = \{S^* \mid S \in \mathcal{F}\}$, with

$$S^* = \begin{cases} S \cup i, & \text{if } S \supseteq K \\ S, & \text{otherwise.} \end{cases}$$

It is convenient to consider * as a mapping from $\mathcal{F}$ to $\mathcal{F}^*$, with $S \mapsto S^*$ defined above. Observe that this mapping is a bijection from $\mathcal{F}$ to $\mathcal{F}^*$, hence the inverse mapping $S^* \mapsto S$ is well-defined.

Consider a balanced collection $\mathcal{B} \subseteq \mathcal{F}$ on N with balancing weights λ_S, $S \in \mathcal{B}$, and its image $\mathcal{B}^*$ by the previous mapping. We claim that $\mathcal{B}^*$ is balanced on N^* if and only if K is either a singleton or a macro-player in $\mathcal{B}^*$, and $(\lambda_{S^*})_{S^* \in \mathcal{B}^*}$ with $\lambda_{S^*} = \lambda_S$ is a system of balancing weights for $\mathcal{B}^*$. Indeed, if K is a macro-player or a singleton, then i is present in $S^* \in \mathcal{B}^*$ if and only if j is present in S^*, for all $j \in K$. It follows that for all $j \in K$,

$$1 = \sum_{S \in \mathcal{B}, S \ni j} \lambda_S = \sum_{S^* \in \mathcal{B}^*, S^* \ni j} \lambda_{S^*} = \sum_{S^* \in \mathcal{B}^*, S^* \ni i} \lambda_{S^*}.$$

Since for all other $j \in N$, we have $\sum_{S^* \in \mathcal{B}^*, S^* \ni j} \lambda_{S^*} = \sum_{S \in \mathcal{B}, S \ni j} \lambda_S = 1$, $\mathcal{B}^*$ is balanced. Conversely, suppose that K is neither a singleton nor a macro-player in $\mathcal{B}^*$. Then there exists $T^* \in \mathcal{B}^*$ such that $j \notin T^* \ni k$, for some $j, k \in K$. Then $i \notin T^*$, so that for every system of positive weights $(\mu_{S^*})_{S^* \in \mathcal{B}^*}$, we have

$$\sum_{S^* \in \mathcal{B}^*, S^* \ni k} \mu_{S^*} > \sum_{S^* \in \mathcal{B}^*, S^* \ni i} \mu_{S^*}. \tag{4}$$

Therefore, no system of balancing weights can exist for $\mathcal{B}^*$.

Conversely, if $\mathcal{B}^* \subseteq \mathcal{F}^*$ is balanced on N^*, the inverse image $\mathcal{B} \subseteq N$ is obviously balanced on N. As a conclusion, the set of balanced collections of $\mathcal{F}^*$ is in bijection with a subset of the balanced collections of $\mathcal{F}$, with same balancing weights. Therefore, defining $v^*_{N^*}$ on $\mathcal{F}^*$ by $v^*_{N^*}(S^*) = v_N(S)$ for all $S \in \mathcal{F}$, $\text{core}(v^*_{N^*}) \neq \emptyset$ if and only if for every balanced collection $\mathcal{B}^*$ on $\mathcal{F}^*$,

$$\sum_{S^* \in \mathcal{B}^*} \lambda_{S^*} v^*_{N^*}(S^*) \leq v^*_{N^*}(N^*)$$

or equivalently,

$$\sum_{S \in \mathcal{B}} \lambda_S v_N(S) \leq v_N(N).$$

We conclude that if the extent core in the context $\mathcal{C}$ is nonempty, then it is also nonempty in $\mathcal{C}^*$. However, there is no guarantee that the converse holds.

We turn to the case of the intent core. Then we must consider the collection $\mathcal{F} = \{M \setminus S \mid S \in L^M_{\mathcal{C}}\}$. Suppose that i is a companion attribute of K, and put $M^* = M \cup i$, and $\mathcal{C}^* = (N, M^*, I^*)$ the new context. Now, $\mathcal{F}^* = \{S^* \mid S \in \mathcal{F}\}$, with

$$S^* = \begin{cases} S \cup i, & \text{if } S \cap K \neq \emptyset \\ S, & \text{otherwise.} \end{cases}$$

As before, * considered as a mapping from $\mathcal{F}$ to $\mathcal{F}^*$ is a bijection. Taking a balanced collection $\mathcal{B} \subseteq \mathcal{F}$ with balancing weights λ_S, $S \in \mathcal{B}$, we claim that $\mathcal{B}^*$ is balanced if and only if K is either a singleton or a macro-attribute in $\mathcal{B}^*$, and weights are identical (proof is similar as before; in the converse part, we have $i \in T$ so that (4) holds with the reverse inequality, replacing k by j). Again, we conclude that the set of balanced collections of $\mathcal{F}^*$ is in bijection with a subset of balanced collections of $\mathcal{F}$, with same balancing weights. Defining $v^*_{M^*}$ on $\mathcal{F}^*$ by $v^*_{M^*}(S^*) = v_M(S)$ for all $S \in \mathcal{F}$, $\text{core}^*(v^*_{M^*}) \neq \emptyset$ iff for every balanced collection $\mathcal{B}^*$ of $\mathcal{F}^*$,

$$\sum_{S^* \in \mathcal{B}^*} \lambda_{S^*} v^*_{M^*}(M^* \setminus S^*) \leq v^*_{M^*}(M^*)$$

or equivalently,

$$\sum_{S\in\mathcal{B}} \lambda_S v_M(M\setminus S) \leq v_M(M).$$

As for the extent core, we conclude that if the intent core in $\mathcal{C}$ is nonempty, it is also nonempty in $\mathcal{C}^*$, with no guarantee for the converse. $\square$

The second question appears to be more tricky. The following example shows that in the general case, there is no clear relation between nonemptiness of the two cores, and any situation may occur.

Example 10 (Example 9 continued). Let us take the concept lattice of Examples 9 and 1, but without specific values for v. The minimal balanced collections for $L^N_{\mathcal{C}}$ are $\{1234\}$ and $\{13, 24\}$, while those for $\overline{L^M_{\mathcal{C}}}$ are $\{abc\}$, $\{bc, ac, ab\}$, $\{c, ab\}$ and $\{bc, a\}$. Hence, by Bondareva-Shapley theorem, the extent core is nonempty if and only

$$v(13) + v(24) \leq v(1234)$$

while the intent core is nonempty if and only if

$$\begin{aligned}\frac{1}{2}v(13) + \frac{1}{2}v(12) + \frac{1}{2}v(24) &\leq v(1234)\\ v(1) + v(24) &\leq v(1234)\\ v(2) + v(13) &\leq v(1234).\end{aligned}$$

The nonemptiness of one of the cores does not imply the nonemptiness of the other one, unless some conditions on v are satisfied. For example, the nonemptiness of the intent core implies the nonemptiness of the extent core if $v(12) \leq v(1234)$, or if $v(1)+v(2) \leq v(1234)$.

Nevertheless, if the concept lattice is distributive, it follows that the extent and intent cores are identical.

Theorem 5. *Suppose that $\mathcal{C} = (N, M, I)$ is an irreducible context, and that $L_{\mathcal{C}}$ is distributive. Then* $\mathrm{core}(v_N)$ *and core*$^*(v_M)$ *are equal.*

The proof relies on the following lemma. We recall that a game $v : L \to \mathbb{R}$ on a lattice $(L, \preceq)$ is *modular* if it satisfies

$$v(x \vee y) + v(x \wedge y) = v(x) + v(y)$$

for every $x, y \in L$.

Lemma 2. *Assume that $(L, \preceq)$ is a distributive lattice with top and bottom elements denoted by $\mathbf{1}, \mathbf{0}$, and v be a game on L. The following statements are equivalent:*

(i) v is modular

(ii) There exists a weight function $w : \mathcal{J}(L) \to \mathbb{R}$ such that for all $x \neq \mathbf{0}$:

$$v(x) = \sum_{p\in\mathcal{J}(x)} w(p) \tag{5}$$

where $\mathcal{J}(x) = \{p \in \mathcal{J}(L) \mid p \preceq x\}$.

(iii) There exists a weight function $w^* : \mathcal{M}(L) \to \mathbb{R}$ such that for all $y \neq \mathbf{1}$:

$$v^*(y) = \sum_{q \in \mathcal{M}(y)} w^*(q) \tag{6}$$

where $\mathcal{M}(y) = \{q \in \mathcal{M}(L) \mid q \succeq y\}$, and $v^*(y) = v(\mathbf{1}) - v(y)$.

Proof. The equivalence between (ii) and (iii) simply comes from the fact that L is distributive if and only if L^∂ is. Therefore it suffices to prove the equivalence between (i) and (ii). Suppose v is given by (5). Then by Birkhoff's theorem, for every $x, y \neq \mathbf{0}$:

$$v(x \vee y) + v(x \wedge y) = \sum_{p \in \mathcal{J}(x) \cup \mathcal{J}(y)} w(p) + \sum_{p \in \mathcal{J}(x) \cap \mathcal{J}(y)} w(p) = \sum_{p \in \mathcal{J}(x)} w(p) + \sum_{p \in \mathcal{J}(y)} w(p) = v(x) + v(y).$$

Conversely, suppose v is modular and define $w(p) = v(p) - v(p')$, where $p' \prec\cdot\, p$, for every $p \in \mathcal{J}(L)$. Assume (by induction on the height of x) that $v(x) = \sum_{s \in \mathcal{J}(x)} w(s)$, and consider some upper neighbor $y \succ\cdot\, x$. By Birkhoff's theorem, there is a unique $p \in \mathcal{J}(L)$ such that $y = x \vee p$, hence the unique lower neighbor p' of p satisfies $x \wedge p = p'$. It follows from modularity that

$$v(y) = v(x) + v(p) - v(x \wedge p) = v(x) + v(p) - v(p') = v(x) + w(p),$$

hence $v(y) = \sum_{s \in \mathcal{J}(y)} w(s)$. □

Proof. (of Theorem 5) The context being irreducible, it follows that N and M correspond to the sets of join-irreducible and meet-irreducible elements of $L_{\mathcal{C}}$, and $M' = \emptyset$. Take an inequality $x(S) \geq v(S, S')$, $S \in L^N_{\mathcal{C}}$ of the extent core, with $x \in \mathbb{R}^N$. Since $N = \mathcal{J}(L^N_{\mathcal{C}})$, it follows from Lemma 2 that x is a weight vector inducing a modular game m on $L_{\mathcal{C}}$. By the Lemma again, this modular game can be written also as a sum of weights on meet-irreducible elements. Denoting by $y \in \mathbb{R}^M$ the weight vector, we have

$$m^*(S, S') = m(N, \emptyset) - m(S, S') = y(S').$$

Since $m(S, S') \geq v(S, S')$ and $m(N, \emptyset) = v(N, \emptyset)$, we obtain

$$y(S') = v(N, \emptyset) - m(S, S') \leq v(N, \emptyset) - v(S, S') = v_M(S'),$$

an inequality of the intent core. Hence, inequalities correspond bijectively and the cores are equal. □

As a consequence, the extent core is nonempty if and only if the intent core is.

4.2 Pointedness and boundedness of the extent core

We assume that $\mathrm{core}(v_N)$ is nonempty. The aim of this section is to study the question whether the core is unbounded and whether it contains a line, in which case it is not pointed (i.e., it has no vertices). The general condition to be pointed is that the system of linear equations

$$x(S) = 0, \forall S \in \mathcal{F}$$

has 0 as its unique solution (in which case we say that, following Derks and Reijnierse [9], $\mathcal{F}$ is *nondegenerate*). It is easy to see that $\mathcal{F}$ is degenerate if there exists a macro-player K in $\mathcal{F}$, because $\mathcal{F}$ contains the hyperplane $x(K) = 0$. A remarkable result with closure systems is that the converse is also true.

Theorem 6. *A closure system is nondegenerate if and only if it contains no macro-player.*

Proof. The "only if" part is obvious since the presence of a macro-player implies degeneracy.

Suppose that $\mathcal{F}$ is a closure system on N with bottom element M', which has no macro-player. We prove by induction on $n = |N|$ that it is nondegenerate. The assertion is easily checked for $n = 1$, with the two possible closure systems $\{\emptyset, \{1\}\}$ and $\{\{1\}\}$. Suppose the assertion holds till some value $n - 1$ and let us prove it for n.

CLAIM: there exists $i \in N$ such that $\{i\} \in \mathcal{F}$.

PROOF OF THE CLAIM: Since $\mathcal{F}$ has no macro-player, we know that its bottom M' is either $\emptyset$ or some singleton. In the latter case, the claim is proved. Suppose then that $M' = \emptyset$. Then necessarily, every atom is a singleton. Indeed, suppose *per contra* that S is an atom, with $|S| > 1$. Since S is not a macro-player, there exists $T \in \mathcal{F}$ separating S, i.e., $j \in T \not\ni k$ for some $j, k \in S$. Since $\mathcal{F}$ is closed under intersection, it follows that $S \cap T \in \mathcal{F}$ and $\emptyset \neq S \cap T \subsetneq S$, a contradiction with the fact that S is an atom. $\square$

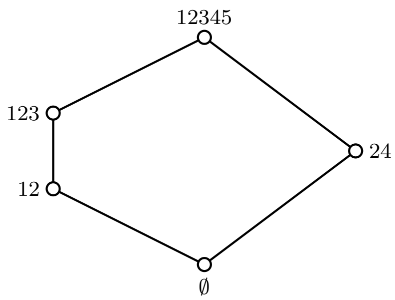
Consider then $\mathcal{F}^{-i} = \{S \subseteq N \setminus i \mid S \text{ or } S \cup i \in \mathcal{F}\}$ on $N \setminus i$, the collection of sets obtained from $\mathcal{F}$ by removing i in every set. Note that $\emptyset \in \mathcal{F}^{-i}$. We prove that $\mathcal{F}^{-i}$ is a closure system without macro-players.

- $\mathcal{F}^{-i} \ni N \setminus i$: clear since $N \in \mathcal{F}$.
- $\mathcal{F}^{-i}$ is closed under intersection: take $S, S' \in \mathcal{F}^{-i}$. Then three cases arise. If $S, S' \in \mathcal{F}$, then $S \cap S' \in \mathcal{F}$ and $i \notin S \cap S'$, hence $S \cap S' \in \mathcal{F}^{-i}$. If $S \in \mathcal{F}$ and $S' \cup i \in \mathcal{F}$, then $i \notin S \cap (S' \cup i) \in \mathcal{F}$, and therefore $S \cap (S' \cup i) = S \cap S' \in \mathcal{F}^{-i}$. Lastly, if $S \cup i, S' \cup i \in \mathcal{F}$, then $i \in (S \cup i) \cap (S' \cup i) \in \mathcal{F}$, therefore $((S \cup i) \cap (S' \cup i)) \setminus i = S \cap S' \in \mathcal{F}^{-i}$.
- $\mathcal{F}^{-i}$ has no macro-player: suppose $K \subseteq N \setminus i$ is a macro-player in $\mathcal{F}^{-i}$. Take $S \in \mathcal{F}^{-i}$. Then either $S \cap K = \emptyset$ or $S \supseteq K$. If $S \in \mathcal{F}$, then $S \cap K = \emptyset$ or $S \supseteq K$ remains true. If $S \cup i \in \mathcal{F}$, then $(S \cup i) \cap K = \emptyset$ or $S \cup i \supseteq K$ is true because $K \not\ni i$. Hence K is a macro-player in $\mathcal{F}$, a contradiction.

Then $\mathcal{F}^{-i}$ is a closure system without macro-player on $N \setminus i$, and by the induction hypothesis, $\mathcal{F}^{-i}$ is nondegenerate, i.e., the system of equations $x(S) = 0, S \in \mathcal{F}^{-i}$ has a unique solution $x = 0$. Finally, observe that the system $x(S) = 0, S \in \mathcal{F}$ differs from the previous one only by the adjunction of x_i in some lines. Since $\{i\} \in \mathcal{F}$, the line $x_i = 0$ makes the two systems equivalent. Therefore, $\mathcal{F}$ is nondegenerate. $\square$

The next example shows that this result does not extend to arbitrary collections of sets.

Example 11. Take $n = 5$ and the collection $\mathcal{F}$ shown below.



$\mathcal{F}$ is not closed under intersection but has no macro-player. However it is degenerate (rank is 4 and $(1,-1,0,1,-1)$ is a vector of the null space).

This being established, we turn to the question whether the core is unbounded or not. The following result proved by Derks and Reijnierse [9] is useful.

Theorem 7. *The recession cone of a game on a collection of sets $\mathcal{F}$ is a linear subspace if and only if $\mathcal{F} \setminus \{\emptyset, N\}$ is a balanced collection.*

Corollary 1. *The core is bounded (equiv., the recession cone reduces to $\{0\}$) if and only if $\mathcal{F}$ is nondegenerate and $\mathcal{F} \setminus \{\emptyset, N\}$ is balanced.*

The situation is summarized by the following table.

	$M' = \emptyset$	$\lvert M'\rvert = 1$	$\lvert M'\rvert > 1$
pointed	if $\mathcal{F}$ has no macro-player	if $\mathcal{F}$ has no macro-player	no
bounded	if $\mathcal{F}$ balanced and no macro-player	no	no

Table 1. Boundedness and pointedness of the extent core

4.3 Some results on balanced collections

For any collection $\mathcal{B} \subseteq 2^N \setminus \{\emptyset\}$, its *closure by intersection* denoted by $\overline{\mathcal{B}}$ is formed by all sets of $\mathcal{B}$, plus the intersection of any family of sets of $\mathcal{B}$, provided the intersection is nonempty[4]. Note that $\overline{(\cdot)}$ is a closure operator, in the sense that $\overline{\overline{\mathcal{B}}} = \overline{\mathcal{B}}$, $\mathcal{B} \subseteq \overline{\mathcal{B}}$, and $\mathcal{B} \subseteq \mathcal{B}'$ implies $\overline{\mathcal{B}} \subseteq \overline{\mathcal{B}'}$.

Theorem 8. *Suppose $\mathcal{B}$ is a balanced collection on N. Then $\overline{\mathcal{B}}$, its closure by intersection, is balanced.*

Proof. Take $\mathcal{B}$ a balanced collection with system of balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. Consider any $S \subseteq N$, and form the collection $\mathcal{B}_S = \{T \cap S : T \in \mathcal{B}\}$. It is easy to see that $\mathcal{B}_S \subseteq 2^S$ is balanced over S with the same system of balancing weights, i.e., the weight of $T \cap S \in \mathcal{B}_S$ is λ_T.

Now, consider $S \in \mathcal{B}$ and the collection $\mathcal{B}_S$. Note that $S \in \mathcal{B}_S$ too. We claim that $\mathcal{B} \cup \mathcal{B}_S$ is a balanced collection over N. Indeed, define the system of weights $(\lambda'_T)_{T\in\mathcal{B}\cup\mathcal{B}_S}$ as follows:

$$\lambda'_T = \begin{cases} \lambda_T, & \text{if } T \in \mathcal{B} \setminus \mathcal{B}_S \\ \frac{\lambda_S}{2}, & \text{if } T = S \\ \frac{\lambda_{T'}\lambda_S}{2}, & \text{if } T \in \mathcal{B}_S \setminus \mathcal{B}, T = S \cap T' \\ \lambda_T + \frac{\lambda_T \lambda_S}{2}, & \text{if } T \in \mathcal{B} \cap \mathcal{B}_S, T \neq S. \end{cases}$$

[4] Be careful that this does not mean that $\overline{\mathcal{B}}$ is closed under intersection, since $\overline{\mathcal{B}}$ does not contain the empty set, despite that it may contain disjoint sets.

Since $\mathcal{B}$ is balanced over N and $\mathcal{B}_S$ is balanced over S, we have for any $i \in S$

$$\sum_{\substack{T \in \mathcal{B} \cup \mathcal{B}_S \\ T \ni i}} \lambda'_T = \sum_{\substack{T \in \mathcal{B}, T \neq S \\ T \ni i}} \lambda_T + \frac{\lambda_S}{2} + \frac{\lambda_S}{2} \sum_{T \in \mathcal{B}_S : T = T' \cap S} \lambda_{T'}$$
$$= \sum_{\substack{T \in \mathcal{B}, T \neq S \\ T \ni i}} \lambda_T + \frac{\lambda_S}{2} + \frac{\lambda_S}{2} = 1.$$

Now, if $i \notin S$, $\sum_{\substack{T \in \mathcal{B} \\ T \ni i}} \lambda'_T = \sum_{\substack{T \in \mathcal{B} \\ T \ni i}} \lambda_T = 1$. Hence, $\mathcal{B} \cup \mathcal{B}_S$ is balanced.

Set $\mathcal{B}' = \mathcal{B} \cup \mathcal{B}_S$. Selecting $T \in \mathcal{B}'$ different from S and computing $\mathcal{B}' \cup \mathcal{B}'_T$ yields another balanced collection on N. Since $\overline{\mathcal{B}}$ is obtained by repeatedly applying this procedure, ultimately $\overline{\mathcal{B}}$ is balanced. □

Lemma 3. *Suppose $\mathcal{B}$ is a balanced collection on N. Then $\overline{\mathcal{B}}$ contains all singletons in N if and only if $\mathcal{B}$ has no macro-player*[5].

Proof. $\Rightarrow$) Clear.

$\Leftarrow$) Suppose $\mathcal{B}$ has no macro-player and for some $i \in N$, $\{i\} \notin \overline{\mathcal{B}}$. Then $\bigcap_{B \in \mathcal{B}, B \ni i} B = S \ni i$, with $|S| > 1$. Since S is not a macro-player, there must exist $T \in \mathcal{B}$ such that $T \not\ni i$ and $T \cap S \neq \emptyset$. Take $j \in T \cap S$, and consider a balancing system $(\lambda_B)_{B \in \mathcal{B}}$ for $\mathcal{B}$. Then

$$1 = \sum_{B \in \mathcal{B}, B \ni i} \lambda_B < \sum_{B \in \mathcal{B}, B \ni i} \lambda_B + \lambda_T \leq \sum_{B \in \mathcal{B}, B \ni j} \lambda_B = 1,$$

a contradiction. □

Note that $\Rightarrow$) holds also if $\mathcal{B}$ is not balanced. An immediate consequence is:

Corollary 2. *If $\mathcal{B}$ is a balanced collection on N, then $\overline{\mathcal{B}}$ contains all macro-players in $\mathcal{B}$ and all singletons in N not contained in macro-players.*

Proof. If there is no macro-player, just apply Lemma 3. Otherwise, replace each macro-player by a single player and apply Lemma 3. □

Remark 2. (i) If $\mathcal{B}$ is minimal balanced and contains no macro-player, then $\mathcal{B} \neq \overline{\mathcal{B}}$. This is clear from Lemma 3 and from the fact that a minimal balanced collection has at most n sets.

(ii) One may wonder if a dual version of Theorem 8 exists, i.e.: if $\mathcal{B}$ is balanced and $\mathcal{B} = \overline{\mathcal{B}}$, then its opening $\mathcal{B}^\circ$ (i.e., removing all sets being intersection of others) is balanced. This is not true as shown by the following example: take $N = \{1, 2, 3, 4\}$ and the balanced collection $\mathcal{B} = \{12, 23, 2, 134, 14, 34\}$ ($\lambda_S = \frac{1}{3}$ can be taken for any $S \in \mathcal{B}$). Its closure is

$$\overline{\mathcal{B}} = \{12, 23, 2, 134, 14, 34, 1, 3, 14, 4\}$$

and is balanced by Theorem 8. Now its opening is $(\overline{\mathcal{B}})^\circ = \{12, 23, 134, 14, 34\}$, but this is not a balanced collection, as it can be checked.

(iii) Observe that in general $\overline{\mathcal{B}_1} \cup \overline{\mathcal{B}_2} \subseteq \overline{\mathcal{B}_1 \cup \mathcal{B}_2}$ with possibly strict inclusion (e.g., take $\mathcal{B}_1 = \{1, 23\}$ and $\mathcal{B}_2 = \{12, 3\}$). It is not sure whether one can obtain any closed balanced collection as a union of the closure of minimal balanced collections.

[5] We mean: there is no macro-player in $\mathcal{B}$. Note that K could be a macro-player in $\mathcal{B}$ but not in $\mathcal{F}$.

4.4 Extremal rays of the extent core

We recall the classical result which holds for the case $\mathcal{F} = \mathcal{O}(N, \preceq)$ (distributive lattice), where $\mathcal{O}(\cdot)$ indicates the set of downsets of some poset.

Lemma 4. *([8, 20]) If $\mathcal{F} = \mathcal{O}(N, \preceq)$, the extremal rays of* $\mathrm{core}(0)$ *are $1^i - 1^j$, for $i \prec\cdot\, j$ in $(N, \preceq)$.*

Let $\mathcal{F} = L_{\mathcal{C}}^N$ be a closure system on N, with bottom element M'. We deal for simplicity with the case where there is no macro-player nor companion player in $\mathcal{F}$ (irreducible closure system). In this case $M' = \emptyset$ and $|\mathcal{J}(\mathcal{F})| = n$, therefore $\mathcal{J}(\mathcal{F})$ can be assimilated to N. We write $(N, \preceq)$ for the poset on N isomorphic to $(\mathcal{J}(\mathcal{F}), \subseteq)$.

Theorem 9. *Suppose that $\mathcal{F}$ is an irreducible closure system. Then $1^i - 1^j$ is a ray of* $\mathrm{core}(\mathcal{F}, 0)$ *for every $i \prec\cdot\, j$ in the poset $(N, \preceq)$, not necessarily extremal. Moreover,* $\mathrm{core}(\mathcal{F}, 0) = \mathrm{core}(\mathcal{O}(N, \preceq), 0)$ *if and only if any $S \in \mathcal{O}(N, \preceq) \setminus \mathcal{F}$ can be written as a union of disjoint sets in $\mathcal{F}$.*

Proof. 1. Let $(N, \preceq)$ be the poset of join-irreducible elements of $\mathcal{F}$. Then $\mathcal{O}(N, \preceq)$ is a distributive lattice and by Lemma 4, $\mathrm{core}(\mathcal{O}(N, \preceq), 0)$ is generated by the rays $1^i - 1^j$, for $i \prec\cdot\, j$ in the poset $(N, \preceq)$. Observe that $\mathcal{F}$ is obtained from $\mathcal{O}(N, \preceq)$ by removing some subsets, therefore $\mathrm{core}(\mathcal{F}, 0) \supseteq \mathrm{core}(\mathcal{O}(N, \preceq), 0)$. Hence any extremal ray of the latter remains a ray in the former, although not necessarily extremal.

2. To prove the second assertion, equality of the recession cones amounts to show that the “missing” inequalities in $\mathrm{core}(\mathcal{F}, 0)$ are implied by the present ones.

2.1. If $S \in \mathcal{O}(N) \setminus \mathcal{F}$ can be written as a disjoint union of sets in $\mathcal{F}$, say $S_1, \ldots, S_k$, then clearly $x(S) \geq 0$ is implied by $x(S_1) \geq 0, \ldots, x(S_k) \geq 0$.

2.2. Conversely, suppose that $x(T) \geq v(T)$ with $T \in \mathcal{O}(N) \setminus \mathcal{F}$ is implied by the other inequalities. It means that there exist $\lambda_S \geq 0$, $S \in \mathcal{F} \setminus \{N\}$, and $\lambda_N \in \mathbb{R}$ such that

$$\sum_{S \in \mathcal{F}, S \neq N, T} \lambda_S 1^S + \lambda_N 1^N = 1^T. \tag{7}$$

2.2.1. Suppose first that for all $\lambda_S > 0$, we have $S \subseteq T$, and $\lambda_N = 0$. Then we have found a subcollection $\mathcal{B} = \{S \in \mathcal{F} : \lambda_S > 0\}$ in $\{S \in \mathcal{F} : S \subseteq T\} =: \mathcal{F}(T)$ which is a balanced collection over T. Since $\mathcal{F}$ is closed under intersection, $\mathcal{F}(T)$ contains $\overline{\mathcal{B}}$. By Lemma 3 and Corollary 2, it follows that $\overline{\mathcal{B}}$ (and hence $\mathcal{F}$) contains all macro-players of $\mathcal{B}$ and all singletons in T. Therefore, we have found a decomposition of T into disjoint sets of $\mathcal{F}$.

2.2.2. Suppose on the contrary that no set of coefficients λ_S, $S \in \mathcal{F}$, satisfies (7) with the condition given in 2.2.1. (i.e., no balanced collection over T exists in $\mathcal{F}$). Then necessarily there exists $S_0 \in \mathcal{F}$ with $\lambda_{S_0} > 0$ which is overlapping T, i.e., $S_0 \setminus T \neq \emptyset$ and $S \cap T \neq \emptyset$. Then (7) forces $\lambda_N < 0$ since for any $j \in S_0 \setminus T$, we have $\sum_{S \in \mathcal{F}, S \neq N} \lambda_S \geq \lambda_{S_0} > 0$.

Let us define $\mathcal{B} = \{S \in \mathcal{F} \mid \lambda_S > 0\}$. Consider $i \in T$. We claim that $\bigcap_{\substack{S \in \mathcal{B} \\ S \ni i}} S \subseteq T$. Otherwise, there would exist $j \in N \setminus T$ belonging to all $S \in \mathcal{B}$ containing i. Then by (7)

$$1 - \lambda_N = \sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \lambda_S \leq \sum_{\substack{S \in \mathcal{B} \\ S \ni j}} \lambda_S = -\lambda_N,$$

a contradiction since $1-\lambda_N > -\lambda_N$. Put $B_i = \bigcap_{\substack{S\in\mathcal{B}\\ S\ni i}} S$. Since $\mathcal{F}$ is closed under intersection, $B_i \in \mathcal{F}$ for every $i \in T$. Observe that if $B_i \ni j$ for some $j \in T$, $j \neq i$, then

$$1-\lambda_N = \sum_{\substack{S\in\mathcal{B}\\ S\ni i}} \lambda_S \leq \sum_{\substack{S\in\mathcal{B}\\ S\ni j}} \lambda_S = 1-\lambda_N,$$

forcing equality throughout. Since all λ_S are positive, it follows that $B_i = B_j$. Discarding equal sets from the collection $(B_i)_{i\in T}$, we have obtained a partition of T with sets in $\mathcal{F}$. But this contradicts the assumption that no balanced collection over T exists in $\mathcal{F}$. $\square$

The above condition is easily violated as shown in the next example.

Example 12. Consider $n=5$ and $\mathcal{F} \subset \mathcal{O}(N,\preceq)$ depicted on Figure 9 together with $(N,\preceq)$ (join-irreducible sets in red). Observe that $\mathcal{F}\setminus\mathcal{O}(N) = \{1234\}$, and that it is not possible

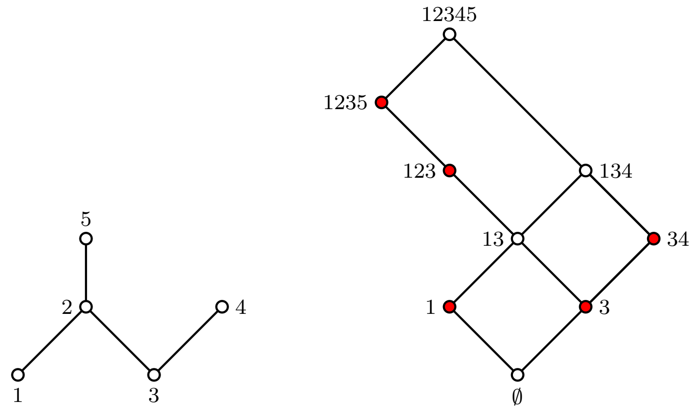


Fig. 9. A lattice (right) with the poset of its join-irreducible elements (left)

to write 1234 as a union of the two atoms 1 and 3. Hence $\mathrm{core}(\mathcal{F},0) \neq \mathrm{core}(\mathcal{O}(N),0)$. This can be verified as $r=(0,0,1,-1,0)$, which is extremal in $\mathrm{core}(\mathcal{O}(N),0)$, is no more an extremal ray of $\mathrm{core}(\mathcal{F},0)$. We can see this in two ways. First, the set of equalities satisfied by r is

$$\begin{aligned} x_1 &= 0 \\ x_3 + x_4 &= 0 \\ x_1 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Observe that the 3d equality is implied by the two first, hence the system determines a 2-dim space, not a ray. Also, it can be checked in the same way that $r_1 = (0,-1,1,-1,1)$ and $r_2 = (0,1,0,0,-1)$ are extremal rays, and that $r = r_1 + r_2$.

4.5 Properties of the intent core

Similar results for the intent core can be obtained easily from the previous results by duality.

Pointedness and boundedness As explained above, and since $\mathrm{core}^*(v_M) = \mathrm{core}(\overline{v}_M)$, all reduces to the study of the system of linear equations

$$y(S) = 0, \quad \forall S \in \overline{L_{\mathcal{C}}^M}.$$

Since $y(S) + y(M \setminus S) = y(M) = 0$ for any $S \in L_{\mathcal{C}}^M$, the above system is equivalent to

$$y(S) = 0, \quad \forall S \in L_{\mathcal{C}}^M.$$

Since $L_{\mathcal{C}}^M$ is closed under intersection, all previous results apply directly. Since $N' = \emptyset$, the situation is simpler than with the extent core, and we find:

(i) The intent core is pointed if and only if $L_{\mathcal{C}}^M$ has no macro-attribute (defined similarly as a macro-player);
(ii) The intent core is bounded if $L_{\mathcal{C}}^M$ is balanced and has no macro-attribute.

Extremal rays We can proceed similarly. The recession cone of the intent core is given by the system

$$\begin{aligned} y(S) &\geq 0, \quad \forall S \in \overline{L_{\mathcal{C}}^M} \\ y(M) &= 0. \end{aligned} \tag{8}$$

Since $y(M) = 0$, proceeding as above, the system is equivalent to

$$\begin{aligned} y(S) &\leq 0, \quad \forall S \in L_{\mathcal{C}}^M \\ y(M) &= 0. \end{aligned} \tag{9}$$

Again, since $L_{\mathcal{C}}^M$ is a closure system, we can benefit from previous results. First, we have the following lemma, similar to Lemma 4.

Lemma 5. *Suppose that $(L_{\mathcal{C}}^M, \subseteq)$ is a distributive lattice, and denote by $(M, \preceq)$ the poset of its join-irreducible elements. Then the extremal rays of $\mathrm{core}^*(v_M)$ are $1^b - 1^a$, for any $a \prec\cdot\, b$ in $(M, \preceq)$.*

Proof. Recall that the intent core is equal to $\mathrm{core}(\overline{L_{\mathcal{C}}^M}, \overline{v}_M)$, hence the recession cone of the intent core is simply $\mathrm{core}(\overline{L_{\mathcal{C}}^M}, 0)$, given by (8). It is equivalent to the system (9), hence $\mathrm{core}(\overline{L_{\mathcal{C}}^M}, 0) = -\mathrm{core}(L_{\mathcal{C}}^M, 0)$. Since $L_{\mathcal{C}}^M$ is a distributive lattice, it is generated by $(M, \preceq)$. It follows from Lemma 4 that extremal rays of $\mathrm{core}(L_{\mathcal{C}}^M, 0)$ are of the form $1^a - 1^b$, with $a \prec\cdot\, b$. Since $\mathrm{core}(\overline{L_{\mathcal{C}}^M}, 0) = -\mathrm{core}(L_{\mathcal{C}}^M, 0)$, the result follows. $\square$

As a consequence, and since $L_{\mathcal{C}}^M$ is a closure system, we obtain by application of Theorem 9 the main result of this section:

Theorem 10. *Suppose that $L_{\mathcal{C}}^M$ is irreducible. Then $1^b - 1^a$ is a ray of $\mathrm{core}^*(0)$ (recession cone of the intent core) for every $a \prec\cdot\, b$ in the poset $(M, \preceq)$, not necessarily extremal. Moreover, $\mathrm{core}^*(0) = \mathrm{core}(\mathcal{O}(M, \preceq^\partial), 0)$ if and only if any $S \in \mathcal{O}(M, \preceq^\partial) \setminus \overline{L_{\mathcal{C}}^M}$ can be written as the intersection of sets in $\overline{L_{\mathcal{C}}^M}$ whose union covers M.*

Note that the poset $(M, \preceq^\partial)$ is isomorphic to the poset of meet-irreducible elements of the concept lattice $L_{\mathcal{C}}$. We formulate the same result in terms of the core of games on dual closure systems.

Corollary 3. *Let $\mathcal{F}$ be a dual closure system on M, and $(M, \preceq)$ the poset of its join-irreducible elements. Then $1^a - 1^b$ is a ray of* $\mathrm{core}(\mathcal{F}, 0)$ *for every $a \prec\cdot\, b$ in $(M, \preceq)$, not necessarily extremal. Moreover,* $\mathrm{core}(\mathcal{F}, 0) = \mathrm{core}(\mathcal{O}(M, \preceq), 0)$ *if and only if any $S \in \mathcal{O}(M, \preceq) \setminus \mathcal{F}$ can be written as the intersection of sets in $\mathcal{F}$ whose union covers M.*

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References

1. E. Algaba, J. M. Bilbao, P. Borm, and J. J. López. The position value for union stable systems. *Math. Meth. Oper. Res.*, 52:221–236, 2000.
2. E. Algaba, J. M. Bilbao, R. van den Brink, and A. Jiménez-Losada. Cooperative games on antimatroids. *Discrete Mathematics*, 282:1–15, 2004.
3. R. J. Aumann and J. H. Drèze. Cooperative games with coalition structures. *Int. J. of Game Theory*, 3:217–237, 1974.
4. J. M. Bilbao, E. Lebrón, and N. Jiménez. The core of games on convex geometries. *European Journal of Operational Research*, 119:365–372, 1999.
5. J.M. Bilbao and P.H. Edelman. The Shapley value on convex geometries. *Discrete Applied Mathematics*, 103:33–40, 2000.
6. N. Caspard, B. Leclerc, and B. Monjardet. *Finite ordered sets — Concepts, results and uses.* Number 144 in Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2012.
7. B. A. Davey and H. A. Priestley. *Introduction to Lattices and Orders.* Cambridge University Press, 1990.
8. J. Derks and R. Gilles. Hierarchical organization structures and constraints on coalition formation. *Int. J. of Game Theory*, 24:147–163, 1995.
9. J.J.M. Derks and H. Reijnierse. On the core of a collection of coalitions. *Int. J. of Game Theory*, 27:451–459, 1998.
10. U. Faigle. Cores of games with restricted cooperation. *ZOR – Methods and Models of Operations Research*, 33:405–422, 1989.
11. U. Faigle and M. Grabisch. A discrete Choquet integral for ordered systems. *Fuzzy Sets and Systems*, 168:3–17, 2011.
12. U. Faigle, M. Grabisch, and M. Heyne. Monge extensions of cooperation and communication structures. *European Journal of Operational Research*, 206:104–110, 2010.
13. U. Faigle and W. Kern. The Shapley value for cooperative games under precedence constraints. *Int. J. of Game Theory*, 21:249–266, 1992.
14. B. Ganter and R. Wille. *Formal Concept Analysis: Mathematical Foundations.* Springer Verlag, 1999.
15. D. Gillies. *Some theorems on n-person games.* PhD thesis, Princeton, New Jersey, 1953.
16. M. Grabisch. The core of games on ordered structures and graphs. *Annals of Operations Research*, 204:33–64, 2013.
17. B. Monjardet. The presence of lattice theory in discrete problems of mathematical social sciences. Why. *Math. Soc. Sci.*, 46:103–144, 2003.
18. R. B. Myerson. Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229, 1977.
19. L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.
20. N. Tomizawa. Theory of hyperspace (XVI)—on the structure of hedrons. Papers of the Technical Group on Circuits and Systems CAS82-172, Inst. of Electronics and Communications Engineers of Japan, 1983. In Japanese.